(2) "Certain features of the application of Makeham's laws of mortality" by Professor H. L. RIETZ, University of Iowa.

(3) "The plan of pensions and insurance recommended by the Carnegie Foundation for the Advancement of Teaching" by Professor E. L. DODD, University of Texas.

(4) Report of progress of the National Committee on Mathematical Requirements, by Professor J. W. YOUNG, Dartmouth College.

(5) "The debt of mathematics to the experimental sciences" by Professor A. C. LUNN, University of Chicago.

(6) Discussion of Professor Lunn's paper with respect to its bearing upon research in pure mathematics, by Professor E. H. MOORE, University of Chicago.

(7) Discussion of Professor Lunn's paper with respect to its bearing on mathematical curricula, by Professor M. W. HASKELL, University of California.

(8) "Retrospect and prospect for mathematics in America,"—retiring presidential address by Professor H. E. SLAUGHT, University of Chicago.

Abstracts of the papers and discussions follow below, the numbers corresponding to the numbers in the list of titles:

1. Professor Campbell's paper dealt with some of the theorems of mathematics underlying the principles of life insurance. He explained various of the principles of interest touching upon nominal and effective interest, force of interest, and present values. Then he took up some problems in probability in connection with the mortality table and derived a few formulas in annuities and insurance, pointing out the special technique of avoiding extensive calculations. He finally treated the subject of reserves and a method of calculating these.

2. In the introduction to this paper, Professor Rietz outlined briefly the historical development of the ideas in Makeham's functions expressive of human mortality. He then gave an exposition of the properties of these functions that make them of fundamental importance in the problems of joint life and survivorship insurance and annuities. It was shown in this paper not only how a Makehamized table following Makeham's first modification of the Law of Gompertz leads to economy of time and energy, but it is shown also how a mortality table following the second modification of the Law of Gompertz may be applied to advantage in the problems of joint life and survivorship insurances and annuities.

3. The Carnegie Foundation recommends, as especially suited to the needs of college teachers, a combination of insurance, non-convertible term insurance or decreasing whole-life insurance, with a savings account—interest at 4 per cent. or $4\frac{1}{2}$ per cent.—to purchase a pension upon retirement or to pass to the estate upon death. These forms of protection—as, indeed, also the more common forms of insurance—can be purchased by teachers from The Teachers Insurance and

Annuity Association at cost. No part of a teacher's payment is used for the expenses of the company.

The more common forms of insurance are (1) non-convertible term, (2) term, convertible to other forms, (3) whole life, (4) limited payment life, and (5) endowment. With the exception of (1), these forms all involve an endowment or cash value, which can be used to purchase a pension. Indeed, with the exception of (1), the forms are all good; and they are well adapted to the needs of a man on moderate income who has some ability to save. For men on very small income, forms (2) and (3) are to be preferred; for men who find it difficult to save, form (5) or the plans suggested by The Carnegie Foundation. The more common forms of insurance are more flexible; but the Foundation plans bind the accumulations strictly for pension purposes. As pension plans, however, they exhibit very considerable flexibility, because of the various options permitted.

4. The report of Professor Young is embodied in the reports printed in the July–September issue of the MONTHLY and elsewhere (pages 441–442) in the present issue.

5. An important portion of the concepts of present mathematical science has emerged by abstraction and generalization from notions originally quite special and concrete occurring in experimental sciences. The successive steps in development have often been carried quite far under the impulse of suggestion from the experimental relations.

Professor Lunn's paper was devoted primarily to a commentary of illustration from the theories of mechanics, heat and electromagnetism. Historical sketches were given of examples leading to familiar general notions in the theories of quadratic forms, modern geometry, differential and integral equations. It is to be hoped that this paper may be available to those not present at the meetings.

6. Professor Moore in his discussion discriminated between the different aspects of research. (1) The process. In research in applied mathematics this does not differ essentially from that in pure mathematics, except that the latter does not need the elaborate equipment necessary in the former. The main speaker, it was pointed out, had brought out the fact that he who comes to pure research with a large background of experimental knowledge has a great aid in his work. Professor Moore instanced by reference to his own study of matrices the possibility of the discovery through generalization of *tools* of great usefulness. (2) Ideals of research, (*a*) of the individual, (*b*) of the group of individuals. Here distinct advantages come directly from the "right" of the experimental sciences to the "left" of the abstract fields; the "foundations" may well serve to make a contribution in the reverse direction. We have much in evidence at the present time in the way of scholarship and of research ability. We must develop the ideal of the group, as is beginning to be done in this country. (3) The form. One should choose the form in which he casts his research such that it shall be most clearly understood by those not conversant with his subject.

There must be a differentiation between various parts of research, yet if science is to advance, there must be a compensatory unifying principle. While mathematical principles have emerged, sometimes directly and logically, sometimes by way of analogy, from the various sciences as described by Dr. Lunn, in the higher reaches of pure mathematics there should be a working out of the principles embodied in physical research in a form free from mere analogy.

7. The mathematical curriculum has changed greatly in recent years. We cannot legitimately include a subject in the curriculum unless we can justify its inclusion to those taking the course. Professor Haskell regarded Professor Lunn's paper as a strong plea for the introduction of the element of interest in mathematics; we must be able to give the student reasons for studying it and must first of all know the reasons ourselves. We must furthermore give the students something in mathematics which shall relate itself to their own lives. We should, for example, show them the advantage of a mathematical formulation; trigonometry has here its great appeal. With less ease, yet in a feasible way, we may make college algebra an attractive subject instead of a bugbear.

Professor Jackson spoke of the lamentable lack of agreement existing between mathematicians and physicists. A single example, he said, would serve, viz., it would aid greatly if we would present existence theorems not so much as proofs of the existence of solutions (which usually are found by entirely different methods) as means of finding solutions where other methods fail.

8. The inspiring retiring address of President Slaught will appear in the December issue of this MONTHLY.

MEETING OF THE COUNCIL OF THE ASSOCIATION.

Eleven members of the Council were present at the meeting.

The following seventy persons and three institutions, on applications duly certified, were elected to membership (making 181 new memberships since January 1, 1920):

NINA M. ALDERTON, A.M. (Columbia). Asst. and grad. student, Univ. of California, Berkeley, Calif.

H. E. ANDERSON, A.M. (Augustana Coll.). Muhlenberg Coll., Allentown, Pa.

W. A. AUSTIN, A.M. (Indiana). Head of dept. of math., High School and Junior Coll., Fresno, Calif.

KATE C. BARBOUR, A.B. (Oklahoma). Teacher, High School, Norman, Okla.

J. F. BARNHILL, A.B. (Kansas). Supt., City Schools, Parsons, Kans.

P. M. BATCHELDER, Ph.D. (Harvard). Instr. in pure math., Univ. of Texas, Austin, Tex.

W. D. BATEN, A.M. (Texas). Head of dept. of math., Grubbs Vocat. Coll., Arlington, Tex.

SÉVÉRIN BAYS, Ph.D. Professeur agrégé, Univ. of Fribourg, Switzerland.

FLORENCE A. BIXBY, A.M. (Columbia). Head of dept. of math., Riverside High School, Milwaukee, Wis.

ETTORE BORTOLOTTI, Dottore in Mat. Prof. ord. di analisi algebrica, Univ. of Bologna, Italy.

M. LUCILE BROWN, A.M. (Ohio State). Instr., Western Coll., Oxford, Ohio.

GLADYS-MARY E. CAMPBELL, A.B. (California). Asst., Univ. of California, Berkeley, Calif.

MICHELE CIPOLLA, Dottore in Mat. (Palermo). Prof. ord. di analisi algebrica, Univ. of Catania, Italy.

R. F. CLARK, A.B. (Williams), Pd.B. (Albany State Normal). Chairman of dept. of math., De Witt Clinton High School, New York, N. Y.

L. H. CUTTING, B.S. (Chicago) Teacher, Westport High School, Kansas City, Mo.

J. A. ELY, C.E. (Princeton). Prof., St. John's Univ., Shanghai, China.

STEPHEN EMERY, A.M. (Boston). Head of dept. of math., Erasmus Hall High School, Brooklyn, N. Y.

P. H. EVANS. Chief actuary, Northwestern Mut. Life Ins. Co., Milwaukee, Wis.

H. L. FASSETT, A.M. (Bucknell). Head of dept. of math., South Side High School, Newark, N. J.

D. D. FELDMAN, B.S. (Nebraska). Prin., Curtis High School, Staten Island, N. Y.

L. R. FORD, Ph.D. (Harvard). Asst. prof., Rice Inst., Houston, Tex.

Z. G. DE GALDEANO, Dr. in Ciencias matemáticas. Prof., Univ. of Zaragoza, Spain.

FALKA M. D. GIBSON, A.B. (California). Teacher, High School, Orland, Calif.

MARIA D. GRAHAM, B.S. (Teachers Coll.). Head of dept. of math., Teachers Training School, Greenville, N. C.

May V. Haworth, Ph.B. (California). Vice-prin., High School, Alameda, Calif.

W. S. HIGGINS, M.E.E. (Harvard). Prof. of math. and engg., Southwestern Presbyt. Univ., Clarksville, Tenn.

L. A. HOPKINS, Ph.D. (Chicago). Asst. prof., Univ. of Michigan, Ann Arbor, Mich.

M. H. INGRAHAM, A.B. (Cornell). Instr., Univ. of Wisconsin, Madison, Wis.

GEORGE JACKSON, B.S. (Cincinnati). Asst. headmaster, Asheville School, Asheville, N. C.

C. L. JOHNSON, B.S. (Ore. Agric. Coll.). Head of dept. of math., Ore. Agric. Coll., Corvallis, Ore.

S. D. KILLAM, Ph.D. (Göttingen). Asso. prof., Univ. of Alberta, Edmonton South, Alb., Canada.

A. V. LEBEUF. Prof. of astr. and dir. of the observatory, Univ. of Besançon, France.

E. J. LEWIS, A.B. (Olivet). Head of dept. of math., Tech. High School, Scranton, Pa.

L. P. LOOMIS, B.S. (Miss. College). Clinton, Miss.

JANE H. MATHEWS, B.S. (Columbia). Teacher, Peabody High School, Pittsburgh, Pa.

L. E. McCarty, A.M. (Texas). Asst. prof., math. and physics, Michigan College of Mines, Houghton, Mich.

Dora McFarland, A.B. (Monmouth). Instr., Univ. of Oklahoma, Norman, Okla.

Elsie J. McFarland, Ph.D. (California). Berkeley, Calif.

Emma L. Noonan, A.M. (Columbia). Teacher, Girls High School, San Francisco, Calif.

J. W. Pancoast, B.S. (Swarthmore). Prof., Guilford College, N. C.

Honor K. Pettit, A.B. (Park Coll.). Grad. student, Univ. of California, Berkeley, Calif.

T. A. Pierce, Ph.D. (California). Asst. prof., Univ. of Nebraska, Lincoln, Neb.

J. F. Pobanz, A.B. (Michigan). Asst., Univ. of California, Berkeley, Calif.

Inez D. Powelson, A.M. (California). Teacher, Bakersfield, Calif.

V. V. Ramana-Sastrin, Ph.D. Vedaraniam, Tanjore Dt., South India.

O. H. Rechard, Jr., A.M. (Penna. Coll.). Instr., Univ. of Wisconsin, Madison, Wis.

Clair Reid, A.B. (Earlham). Instr., Purdue Univ., LaFayette, Ind.

Theresa M. Renner, B.S. (Blackburn). Instr., Blackburn Coll., Carlinville, Ill.

C. P. Rockwell. Asst. actuary, Texas Dept. of Ins. and Banking, Austin, Tex.

Sarah A. Ruby, A.B. (Iowa). Head of dept. of math., Jefferson High School, Portland, Ore.

G. O. Sagen, A.B. (California). Asst., Univ. of California, Berkeley, Calif.

Meyer Salkover, A.M. (Cincinnati). Instr., Univ. of Cincinnati, Cincinnati, Ohio.

Minna J. Schick, A.M. (Northwestern). Instr., Univ. of Minnesota, Minneapolis, Minn.

L. Silberstein, Ph.D. Research laboratory, Eastman Kodak Co., Rochester, N. Y.

E. B. Skinner, Ph.D. (Chicago). Prof., Univ. of Wisconsin, Madison, Wis.

I. W. Smith, A.M. (Illinois). Prof., North Dakota Agric. Coll., Fargo, N. D.

W. A. Stafford, A.M. (Stanford). Head of dept. of math., High School, Oakland, Calif.

G. C. Staley, A.M. (Chicago). Instr., Parker High School, Chicago, Ill.

Stella Stephens, A.B. (Georgetown Coll.). Teacher, High School, Paris, Ky.

Helen Thompson, A.B. (Vassar). Head of dept. of math., Kentucky Coll. for Women, Danville, Ky.

Luis Octavio de Toledo. Prof., Univ. of Madrid, Spain.

R. S. Underwood, A.M. (Minnesota). Instr., Purdue Univ., LaFayette, Ind.

Louis Van Hee. Jesuit Father; Prof of math., Liége, Belgium.

P. W. Waterman, Ph.B. (Vermont). Head of dept. of math., Montclair Acad., Montclair, N. J.

J. H. M. Wedderburn, D.Sc. (Edinburgh). Asst. prof., Princeton Univ., Princeton, N. J.

Helen F. Weeks, B.S. (California). Head of dept. of sc. and math., High School, Alhambra, Calif.

B. C. WONG, A.M. (California). Asst., Univ. of California, Berkeley, Calif.
RUTH G. WOOD, Ph.D. (Yale). Prof., Smith Coll., Northampton, Mass.
JESSICA M. YOUNG, M.S. (California). Instr., math. and astr., Washington Univ., St. Louis, Mo.
VIVIAN YOUNG, A.B. (Willamette). Head of dept. of math., High School, Salem, Ore.

To institutional membership.

ST. MARY-OF-THE-WOODS COLLEGE, St. Mary-of-the-Woods, Ind.
NORTHERN NORMAL AND INDUSTRIAL SCHOOL, Aberdeen, S. Dak.
UNIVERSITY OF WYOMING, Laramie, Wyo.

A report was made for the committee on life membership. It was agreed that more study should be devoted to this subject and a committee of three actuarial members of the Association was appointed which should report on the actuarial features of the proposed life membership fee, including the feasibility of a fee graduated according to age.

It was voted to accept the invitation of Wellesley College for the summer meeting in 1921.

It was voted to accept the invitation of the staff of the University of Chicago for the next annual meeting in affiliation with the meeting of the American Association for the Advancement of Science; and through a later report of the committees of the Association and the Society it was agreed that there should be morning and afternoon sessions of the Association on Tuesday, December 28, simultaneous sessions of the two bodies on Wednesday morning, a joint session of these with Section A of the American Association on Wednesday afternoon, and further sessions of the Society on Thursday.

The most important action of the Council was in regard to the financial situation of the Association. Like other scientific periodicals, the cost of publication of the MONTHLY has increased about 50 per cent. during the past year; and provision needs now to be made for editorial and clerical assistance in the office of the editor-in-chief. Because of these additions, it appears, according to a report made to the Council by the Secretary-Treasurer, that the reserve in the Association treasury will be more than wiped out in the next year if no additional income is secured. The Council after careful and full consideration, both by correspondence during the past summer and by conference at the Chicago meeting, believes that the members are so strongly committed to the ideals and accomplishments of the Association and of its worth to themselves, that they will loyally approve increased membership dues as one means of offsetting the increased expenses. The Council therefore voted to increase the individual dues to four dollars, the institutional dues to seven dollars, and subscriptions for non-members to five dollars, beginning with January, 1921. An exception was made to this in the case of those who at that time will have been members for less than one year; it seems only fair to exempt these from the increase until January, 1922, thus giving them a full year on the basis of the dues prevailing when they joined.

Since twice as much additional revenue must be secured as will come from the increase in membership dues, if the income of the Association is to equal its expenditures, a committee of the Council has been appointed to invite subscriptions, aside from membership dues, which may be applied to the current budget or to permanent endowment. A third method by which members can readily come to the financial aid of the Association is for each member to invite and urge others to become members; if we believe in the value of the Association's work, we can rightfully bring this to the attention of all the members of the staff and to advanced students of mathematics at each college and university, and to other persons definitely interested in mathematics.

To the end that our organization may legally receive donations and bequests, the Council took steps to incorporate the Association. Both the Council and the Association unanimously adopted resolutions empowering Professors E. H. Moore, H. E. Slaught and W. D. Cairns to apply for a charter, and this action was taken by them at once. Since then the charter has been granted and the Mathematical Association of America Incorporated has been organized under the statutes of the State of Illinois. The articles of incorporation vest the legal control in the first instance in three trustees, Professors H. E. Slaught, E. R. Hedrick and W. D. Cairns, but they have exercised their legal prerogative by enlarging the Board of Trustees to nineteen, thus including the present officers and all members of the present Council and by adopting a set of by-laws which include the by-laws and constitution of the original Association together with such modifications and additions as were necessary to meet the legal requirements. The charter of the Mathematical Association of America Incorporated now supersedes the old constitution. The property and assets of the old Association have been transferred to the newly incorporated body in accordance with the authority granted by vote of the Council. The By-Laws of the Mathematical Association of America Incorporated follow, preceded by the official minutes of the organization meeting and the Certificate of Incorporation.

W. D. CAIRNS, *Secretary-Treasurer.*

CERTIFICATE NO. 3651.

STATE OF ILLINOIS, OFFICE OF SECRETARY OF STATE.

To all to whom these Presents shall come, Greeting:

Whereas a certificate, duly signed and acknowledged has been filed in the office of the Secretary of state, on the 8th day of September, A.D. 1920, for the organization of

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

under and in accordance with the provisions of "An Act Concerning Corporations" approved April 18, 1872, and in force July 1, 1872 and all acts amendatory thereof, a copy of which certificate is hereto attached;

Now, therefore, I, Louis L. Emmerson, Secretary of State of the State of Illinois, by virtue of the powers and duties vested in me by law, do hereby certify that the said

THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

is a legally organized corporation under the laws of this State.

IN TESTIMONY WHEREOF I hereunto set my hand and cause to be affixed the Great Seal of the State of Illinois done at City of Springfield, this 8th day of September A.D. 1920 and of the Independence of the United States the one hundred and forty-fifth.

LOUIS L. EMMERSON,
Secretary of State.

[GREAT SEAL OF THE STATE OF ILLINOIS]

[Filed for Record September 10, 1920 in the office of the Recorder of Cook County, Illinois as Document No. 6935590.]

MINUTES OF THE ORGANIZATION MEETING OF THE TRUSTEES OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED), HELD AT CHICAGO, ILLINOIS, ON THE 10TH DAY OF SEPTEMBER, A. D. 1920.

There were present Messrs. Slaught, Hedrick and Cairns, being all of the Trustees acting by virtue of Application for Charter filed with the Secretary of State at Springfield, Illinois.

The meeting was called to order by Mr. Slaught, who presided. Mr. Cairns acted as Secretary of the meeting and recorded the proceedings.

Mr. Slaught reported that Application for Charter had been duly filed with the Secretary of State, that Certificate of Organization had thereon been issued by the Secretary of State and that such Certificate had been filed with, and on this 10th day of September, A. D. 1920, recorded in the office of, the Recorder of Deeds of Cook County, all in accordance with the laws of the State of Illinois; that the Association had, under the law, been fully organized and might now proceed to business.

The Chairman announced that at a meeting of the members of The Mathematical Association of America, the unincorporated organization to which it is proposed to make this Association successor, held at Chicago, Illinois, September 6, 1920, a resolution had been adopted, authorizing the Officers and the members of the Council of that organization to do all acts and things necessary to be done to complete the organization of this Association, to transfer to it the property and business of the unincorporated organization, subject to its liabilities, and, so far as possible, to provide for the continuation of the Association's business and affairs substantially along the lines prevailing in the unincorporated organization.

The Chairman announced that the first order of business was the election of a President and Secretary of the newly incorporated Association.

Thereupon, on motion duly made, seconded and unanimously carried, it was resolved that until, and subject to, the legal effectuation of the amendment to the Articles of Association for the increase of Trustees, as contemplated and hereinafter set forth, HERBERT E. SLAUGHT and WILLIAM D. CAIRNS be, and hereby are, elected, respectively, President and Secretary of this Association.

The Chairman announced that the next order of business was the adoption of By-Laws for the government of the Association and thereupon submitted a code of By-Laws which was carefully considered and, upon motion duly made, seconded and unanimously carried, adopted as the By-Laws of this Association. A copy of the By-Laws so submitted and adopted was ordered certified by the Chairman of the meeting and included in the records of this meeting.

Thereupon, on motion duly made, seconded and unanimously carried, it was resolved that Section 3 of the Articles of Association of THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED) be, and hereby is, changed and amended to read as follows: "The management of the aforesaid Association shall be vested in a Board of Nineteen (19) Trustees."

Thereupon, on motion duly made, seconded and unanimously carried, it was resolved that the Officers of this Association be, and hereby are, authorized and directed to do all acts and things necessary, proper or convenient to be done in and about making legally effectual the amendment of the Articles of Association aforesaid and the consummation of the transfer to this Association of the assets, property, business and membership of the present unincorporated Association bearing the same name, including payment of all fees and expenses incident thereto out of the funds of the Association.

No further business being presented, the meeting was, on motion, duly adjourned.

WILLIAM D. CAIRNS,
Secretary of the Meeting.

Approved:

HERBERT E. SLAUGHT,
Chairman of the Meeting.

BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED).

ARTICLE I—NAME, PURPOSE AND CORPORATE SEAL.

1. This organization shall be known as

THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED).

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by cooperating with other organizations whenever this may be desirable for attaining these or other similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

ARTICLE II—MEMBERSHIP.

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association. Such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

3. Election to membership shall be by vote of the Board upon written application from the individual or institution seeking admission.

4. Those who were admitted to membership in The Mathematical Association of America (unincorporated) prior to October 1, 1920, and are in good standing as such are hereby admitted to membership in this Association.

ARTICLE III—BOARD OF TRUSTEES AND OFFICERS.

1. The control and management of the affairs and funds of the Association shall be vested in a Board of Trustees, who shall be members of the Association. The Board of Trustees, in the first instance, shall consist of the three (3) persons named in the Certificate of Organization. It is contemplated that such Board of Trustees shall, immediately after completing the organization of the Association, amend the Articles of Association, in the manner hereinafter prescribed, to provide for a Board of Trustees numbering Nineteen (19). Such Board of Nineteen (19) shall consist of a President, two (2) Vice-Presidents, a Secretary-Treasurer, three (3) members of the Committee on Publications and twelve (12) additional members.

2. For the terms set against their respective names, the following shall, upon amendment of the Articles of Association above mentioned, be the Officers and Trustees of the Association:

Office.	Name.	Term ending with the Annual Meeting in
President	D. E. Smith	December, 1920 or January, 1921.
Vice-President	Helen A. Merrill	December, 1920 or January, 1921.
Vice-President	E. J. Wilczynski	December, 1920 or January, 1921.
Secretary-Treasurer	W. D. Cairns	December, 1920 or January, 1921.
Committee on Publication	W. A. Hurwitz, H. E. Slaught, *Manager*, R. C. Archibald, *Editor in Chief*	December, 1920 or January, 1921.
Trustees	Florian Cajori Elizabeth B. Cowley E. L. Dodd G. A. Miller	December, 1920 or January, 1921.
Trustees	L. P. Eisenhart B. F. Finkel E. V. Huntington E. H. Moore	December, 1921 or January, 1922.
Trustees	R. D. Carmichael E. R. Hedrick Oswald Veblen J. W. Young	December, 1922 or January, 1923.

3. The President and Vice-Presidents shall be elected by the Association's members annually for a term of one year, and four members of the Board shall be elected by the Association's members annually for a term of three years. They shall be eligible for reelection, but not for more than two (2) consecutive terms. The Secretary-Treasurer, and the Committee on Publications, consisting of the Manager, the Editor and one other member, shall be appointed by the Board. All Trustees and Officers shall hold over until their respective successors are elected and qualify.

4. The Board shall transact the official business of the Association and shall report its actions at the annual meeting of the Association and in the official journal. Any proposed action of the Board which makes or alters a question of policy shall be published in the official journal before final action has been taken, so that members of the Association may make known to the Board their individual views.

5. The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board and in the Committee on Publications.

8. At all meetings of the Board of Trustees a quorum shall consist of not less than five (5) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Board, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement at such meeting. Informal action based on correspondence among the members of the Board, if ratified at a properly convened meeting of the Board, shall be as valid and effective as if originally authorized at such meeting.

7. Two months before the date of the annual meeting all members shall be given an opportunity to nominate by mail a candidate for each office to be elected by the members for the ensuing year. One month before the annual meeting the Board shall announce two candidates for each office to be filled by the members, one being the person who received the highest vote in the nominations and the other being selected by the Board from among the several nominees next in order. The election shall be by mail or in person and shall close on the day of the annual meeting.

8. The President shall be the executive Officer of the Association, shall preside at all meetings of the Board of Trustees and at the annual meeting of the Association. He shall have the usual duties pertaining to his office and such other duties as may from time to time be devolved upon him by the Board of Trustees.

9. The Vice-Presidents shall, in the absence of the President, have and exercise the powers of the President, their order being determined alphabetically. The Board of Trustees may devolve upon the Vice-Presidents such duties as may from time to time be determined.

10. The Secretary-Treasurer shall have the usual duties pertaining to the office of Secretary and of Treasurer, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Trustees and of the annual meeting and special meetings of members, the giving of due notice of all regular and special meetings of the Association and of the Board of Trustees and the supervision and safe-keeping of the funds

of the Association. The Secretary-Treasurer shall also have the duty of seeing that whenever Trustees are elected, including the election of Trustees to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the term of their office, shall be recorded in the office of the Recorder of Deeds for Cook County, Illinois. Such Certificate shall be signed by the Secretary-Treasurer and verified by oath of the President.

11. The Committee on Publications shall have supervision, subject to the control of the Board of Trustees, of the official journal and of all other publications of the Association.

Article IV—Meetings.

1. A meeting of the Association shall be held annually, at such time and place as the Board may direct. Special meetings of the Association may be called from time to time by the Board of Trustees, or while the Board is not in session by the President of the Association, to be held at such time and place as may appear from the call. The first annual meeting of the Association shall be held in December, 1920 or January, 1921.

2. The outgoing Board of Trustees shall hold a meeting immediately preceding the annual meeting of the Association next succeeding their election, and the members of the new Board of Trustees shall hold a meeting and organize, by completing the Board, immediately succeeding the annual meeting of the Association at which the new members thereof were elected. Further meetings of the Board may be held from time to time at the call of the President or any three (3) members of the Board.

3. Notice of any meeting of members of the Association shall be given by the Secretary-Treasurer at least thirty (30) days prior to the date set for such meeting. Notice of all meetings of the Board of Trustees other than the regular meetings provided in Section 2 shall be given to each member of the Board at least fifteen (15) days prior to the date set therefor.

4. Any member of the Association or of the Board of Trustees may waive notice with the same effect as if due notice had been given him.

5. At all meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Association, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement at such meeting.

6. Members may take part and vote in person or by proxy at all meetings of the Association.

Article V—Sections.

1. Any group of members of this Association may petition the Board for authority to organize a Section of the Association for the purpose of holding local meetings. The Board shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such Sections.

Article VI—Official Organ.

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Board shall have full control of the publication and sale of the official journal.

3. The official journal shall be under the general management of the Committee on Publications. There shall also be appointed by the Board a body of Associate Editors who shall give assistance in connection with the official journal and under the direction of the Committee on Publications. The Board may appoint any other committees and delegate to them such power as may, in its judgment, seem desirable.

4. The Board shall fix the price of the official journal and of any other publications of the Association to non-members, but in no case shall the journal be sold for less than the annual dues of individual members.

Article VII—Dues.

1. Individual members of the Association shall each pay an initiation fee of Two Dollars ($2) at the time of election.

2. The annual dues of each individual member shall be Four Dollars ($4), including a subscription to the official journal.

3. The annual dues of each institutional member shall be Seven Dollars ($7), including two (2) subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list, after due notice.

5. New members entering the Association after April 1 of any year shall have their dues pro rated for the balance of the year, except when they desire to receive the full current volume of the official journal.

ARTICLE VIII—AMENDMENTS TO THE ARTICLES OF ASSOCIATION AND BY-LAWS.

1. The Articles of Association may be changed to provide for a Board of Nineteen (19) Trustees by vote of a majority of the Three (3) Trustees named in the original Articles. Subsequent changes, amendments or modifications, of the Articles of Association and any amendments to the By-Laws may be made at any annual meeting of the Association, or at any adjourned session thereof, or at any special meeting of the Association called for such purpose, by a two-thirds (2/3) vote of those present and entitled to vote; *provided* that such amendment shall have been printed in the official journal at least one (1) month before the date of such meeting.

2. No change in the Articles of Association shall have legal effect until a Certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary-Treasurer, shall be filed in the office of the Secretary of State of the State of Illinois and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

ARTICLE IX—INTERPRETATION.

1. "Board," wherever used in these By-Laws, shall be taken to mean a Board of Trustees consisting of the Officers, the Committee on Publications and the Trustees elected as such, as provided in Article III.

2. It being the intent that this Association continue as successor to the unincorporated The Mathematical Association of America, these By-Laws shall be construed liberally to that effect.

THE UNDERSIGNED, CHAIRMAN OF THE ORGANIZATION MEETING OF THE BOARD OF TRUSTEES OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED), HEREBY CERTIFIES that the above is a full, true and correct copy of the code duly adopted at such meeting, held September 10, 1920, as the By-Laws of said Association.

WITNESS MY HAND AND THE CORPORATE SEAL, this 11th day of September, A. D. 1920.

[CORPORATE SEAL] HERBERT E. SLAUGHT,
Chairman as aforesaid.

THE "DANGER AREA" CURVE.

By A. S. MERRILL, University of Montana.

A moving object (ship) is a target for a projectile (torpedo) which travels horizontally under its own power. Certain limiting conditions under which the projectile would be fired, or launched, are known. It is desired to determine the boundary curve of the area of probable positions of the firing agent relative to the target, at the time of firing.

Let the target, of length AB but of negligible width, be considered first as fixed in position. For certain known conditions of probable error, the projectile has a range R if fired from a point in the perpendicular bisector of AB. Construct a circle of diameter R through A and B (Fig. 1). Suppose AB to be small in comparison with R. The middle point of AB is then approximately at M, the end point of the diameter ML perpendicular to AB,—sufficiently close to be

so considered for our problem. If the target is viewed from any other point upon the circle, as C, it is evident that it subtends an angle of the same size as that which it subtends when viewed from L. The probability of a hit is then the same for a projectile launched from any point upon the circle. In other words, the circle is the locus of points from which the projectile may be fired with the same probability of success for the fixed limiting conditions of probable error, and is the boundary curve of the area from which the projectile may be fired within the limits of probable success.

If now the target is considered as moving, there arise very difficult complications in the matter of aiming the projectile. With these, however, we are not at present concerned. For our problem the difficulties are not great.

Let the target be moving in the direction from A to B at the rate r, while the speed of the projectile is s. While the projectile moves a certain distance, the target moves r/s times this distance from a former position to the position AB. For projectiles fired from equal distances, the target, of course, moves through the same distance. The following scheme then enables us to plot very accurately the desired curve.

Construct Fig. 2 on tracing paper as follows. On the circle of Fig. 1 construct the tangents MK and LQ at M and L respectively. With M as a center and

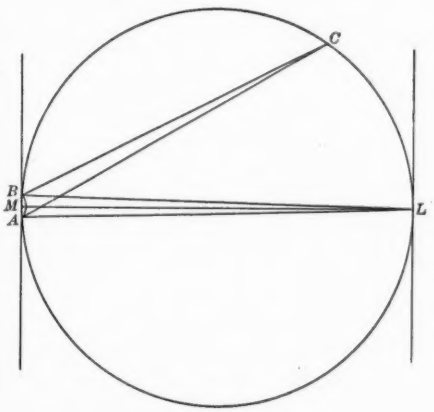


FIG. 1.

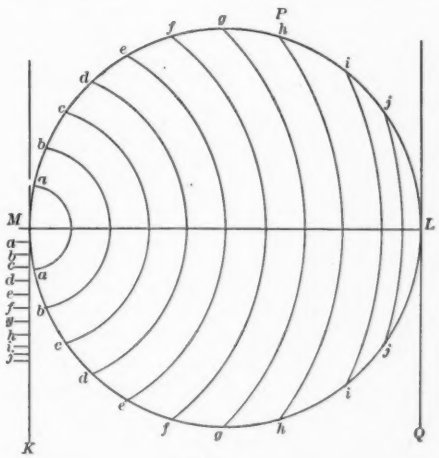


FIG. 2.

radii of lengths ranging at convenient intervals from 0 to R, strike arcs cutting the circle. From M toward K measure off distances equal to r/s times the lengths of these radii. Mark these points of division with the same symbols as the points of intersection of the corresponding arcs with the circle. On drawing paper draw two parallel lines at a distance R from each other, as OY and XV of Fig. 3. Place Fig. 2 over this drawing paper and, keeping MK and LQ coincident with OY and XV respectively, place the different division points of MK

successively over O. At each position, punch through the two papers with a pin point at the two points of the circle marked correspondingly. Draw a smooth curve (dotted in Fig. 3) through the points thus marked on the drawing paper. This is the required curve.

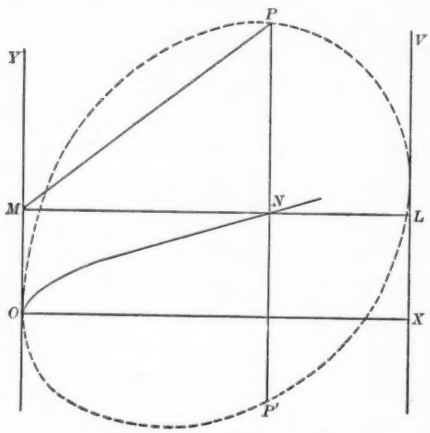


FIG. 3.

The area within the curve of Fig. 3 is referred to as the "danger area." It is evident that similar danger areas exist on both sides of AB, reflections of each other with respect to that line.

The above device is due to an English mathematician, and was used in naval strategical study during the late war. It has been interesting to work out the equations of a curve which has such an application. Results from two different simple methods are given below.

I. Let P be a point of the circle of Fig. 2 and the point determined from it in Fig. 3. Let $\angle LMP = \theta$. Then $MP = ML \cos\theta = R \cos\theta$, and

$$NP = MP \sin\theta = R \cos\theta \sin\theta.$$

Taking O as the origin, OX and OY as axes, we have

$$x = MN,$$

$$y = OM \pm NP \text{ (since } NP' = -NP).$$

But $MN = MP \cdot \cos\theta = R \cdot \cos^2\theta$, and $OM = r/s \cdot R \cdot \cos\theta$. Hence we have the parametric equations of the curve:

$$x = R \cos^2\theta,$$

$$y = R \cos\theta \cdot ((r/s) + \sin\theta).$$

Whence we have

$$\cos\theta = \sqrt{x/R} = (1/R)\sqrt{Rx}, \qquad \sin\theta = \pm (1/R)\sqrt{R^2 - Rx};$$

and then

$$y = (r/s)\sqrt{Rx} \pm \sqrt{Rx - x^2}$$

is the equation of the curve.

II. The curve of Fig. 3 is the locus of points of intersection of corresponding curves of two families of circles. Of one of these families, the curves all have the same radius $R/2$, but their centers move along a line parallel to the line OY (Fig. 3) and midway between lines OY and XV. The radii, ρ, of the circles of

the second family vary from 0 to R, and the centers are along OY. The centers of corresponding curves are at the same distance above O, and the radius of a circle of the second family is equal to s/r times this distance. The equations of these families are then respectively:

$$(x - R/2)^2 + [y - (r/s)\cdot\rho]^2 = R^2/4, \tag{1}$$

$$x^2 + [y - (r/s)\cdot\rho]^2 = \rho^2. \tag{2}$$

Subtracting (1) from (2) we have

$$Rx - R^2/4 = \rho^2 - R^2/4,$$

or

$$\rho^2 = Rx.$$

Substituting in (2) we obtain

$$x^2 + [y - (r/s)\sqrt{Rx}]^2 = Rx,$$

$$y - (r/s)\sqrt{Rx} = \pm\sqrt{Rx - x^2},$$

or

$$y = (r/s)\sqrt{Rx} \pm \sqrt{Rx - x^2},$$

which is the equation already obtained in I.

The general shape of this curve is evident from an inspection of this equation. The locus of the equation

$$y = (r/s)\sqrt{Rx}$$

is evidently the upper branch of the parabola

$$y^2 = (r/s)^2\cdot Rx.$$

The quantity $\sqrt{Rx - x^2}$ is real for positive values of x less than R, and is zero for $x = 0$ or $x = R$. The danger area curve is thus seen to be a loop lying between the lines $x = 0$ and $x = R$. The points of this loop are at the distance $\sqrt{Rx - x^2}$ above and below the points (x, y) of the upper branch of the parabola $y^2 = (r/s)^2 Rx$. If the objective had been considered as moving in the opposite direction, the danger area loop would have borne the same relation to the lower branch of this same curve. The danger area on the opposite side of the objective is similarly related to the parabola $y^2 = -(r/s)^2\cdot Rx$.

EXPRESSIONS FOR CERTAIN ACCELERATIONS OF A PARTICLE.

By GEORGE H. CRESSE, U. S. Naval Academy.

The object of this paper is to present a method of obtaining expressions for the magnitudes of certain accelerations of a particle in terms of its speed and from our knowledge of its path. Let the path be plane and considered only at non-singular points, and let its equation be $f(x, y) = 0$. By differentiation with respect to time, we see that the motion of the particle satisfies the equation,

$$f_x'x' + f_y'y' = 0 \tag{1}$$

where x', y' are time-derivatives. By differentiating (1) with respect to time and dividing by $\sqrt{f_x'^2 + f_y'^2}$, we have,

$$\frac{f_x''x'^2 + 2f_{xy}''x'y' + f_y''y'^2}{\sqrt{f_x'^2 + f_y'^2}} + \frac{f_x'x'' + f_y'y''}{\sqrt{f_x'^2 + f_y'^2}} = 0. \tag{2}$$

where x'', y'' are second time-derivatives. In the light of (1), we may put

$$\frac{f_x'x'' + f_y'y''}{\sqrt{f_x'^2 + f_y'^2}} = \frac{-y'x'' + x'y''}{\sqrt{x'^2 + y'^2}} = \alpha \cos\theta,$$

where α is the magnitude of the acceleration, and θ is the angle between the normal to the curve and the direction of acceleration. Hence for (2), we may write,

$$\frac{f_x''x'^2 + 2f_{xy}''x'y' + f_y''y'^2}{\sqrt{f_x'^2 + f_y'^2}} = -\alpha \cos\theta. \tag{3}$$

I. For example, let $f(x, y) = x^2 + y^2 - r^2$, and let the acceleration be directed toward the center of the circle. (3) now becomes the familiar formula:

$$\frac{x'^2 + y'^2}{\sqrt{x^2 + y^2}} = \alpha, \quad \text{or} \quad \alpha = \frac{v^2}{r}. \tag{4}$$

II. For another example, let $f(x, y) = b^2x^2 + a^2y^2 - a^2b^2$, and let the acceleration be directed toward one focus F of the ellipse. For this example, (3) is

$$\frac{b^2x'^2 + a^2y'^2}{\sqrt{b^4x^2 + a^4y^2}} = -\alpha \cos\theta. \tag{5}$$

If r and r' be the focal distances from F and F' respectively to the particle at P, and p and p' be perpendiculars drawn from the foci to the tangent which has

(Read at the meeting of the Maryland-Virginia-District of Columbia meeting of the Mathematical Association of America, May 15, 1920.)

P for its point of tangency, elementary properties of the ellipse are that,

$$\cos\theta = -\frac{p}{r} = -\frac{p'}{r'} = -\sqrt{\frac{pp'}{rr'}} = -\frac{b}{\sqrt{rr'}} = \frac{-b}{\sqrt{a^2-e^2x^2}} = \frac{-ab^2}{\sqrt{b^4x^2+a^4y^2}}.$$

And, since on the ellipse $b^2xx' + a^2yy' = 0$, the numerator in the first member of (5) has the value

$$\frac{b^4x'^2}{y^2} = \frac{a^4y'^2}{x^2}.$$

Hence (5) implies

$$(6_1)\quad x'^2 = \frac{a\alpha y^2}{b^2}, \qquad (6_2)\quad y'^2 = \frac{b^2\alpha x^2}{a^3};$$

whence

$$v^2 = \frac{\alpha(a^4y^2 + b^4x^2)}{a^3b^2} = \frac{\alpha rr'}{a},$$

or

$$(7)\qquad \alpha = \frac{av^2}{rr'},$$

which is a generalization of (4).

By differentiating (6_1) as to time, and setting in the value of y' from (6_2), we have,

$$x'' = \frac{1}{2a}\frac{d\alpha}{dx}(a^2 - x^2) - \frac{x\alpha}{a}.$$

But the x-component of acceleration is

$$x'' = \alpha\frac{ae - x}{a - ex}.$$

From the comparison of the last two equations, we have

$$\frac{d\alpha}{2\alpha} = \frac{e\,dx}{a - ex}.$$

Whence the Newtonian law:

$$(8)\qquad \alpha = \frac{k^2}{(a - ex)^2} = \frac{k^2}{r^2}.$$

A comparison of (7) and (8) yields the following elegant form of the ***vis viva*** equation:

$$v^2 = \frac{k^2}{a}\frac{r'}{r}.$$

The above discussion of elliptical motion, with only the slightest modification is valid for hyperbolic and parabolic motion under a central force directed toward a focus. And the last equation is valid in all three cases without modification, r'/a having the limiting value 2 in the case of the parabola.

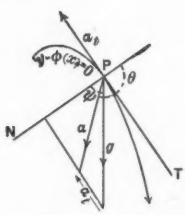
III. For the path of a projectile, we may take a parabola of higher order and let

$$f(x, y) = y - (a_0 + a_1x + a_2x^2 + \ldots + a_nx^n) = y - \varphi(x).$$

The range is assumed so short that the vertical direction may be regarded as constant. Now α is the resultant of the vertical gravitational acceleration $-g(y)$, and a tangential acceleration α_t. Obviously $\alpha \cos\theta = -g\cos\psi$, where ψ is the angle between the normal and the vertical. Hence (3) for this example is, by reference to (1),

$$-\frac{\varphi''(x)x'^2}{\sqrt{f_x'^2+f_y'^2}} = g\cos\psi = g\frac{dx}{\sqrt{dx^2+dy^2}} = g\frac{f_y'}{\sqrt{f_x'^2+f_y'^2}}.$$

or,



$$x' = \frac{\sqrt{g}}{[-\varphi''(x)]^{1/2}}.$$

Therefore,

(9) $$v = \frac{\sqrt{g}}{[-\varphi''(x)]^{1/2}}\sqrt{1+\left(\frac{dy}{dx}\right)^2};$$

and,

$$x'' = \frac{g}{2}\frac{\varphi'''(x)}{[\varphi''(x)]^2}.$$

But since the total horizontal acceleration is the horizontal component of α_t, it follows that

(10) $$\alpha_t = \frac{g\varphi'''(x)}{2[\varphi''(x)]^2}\cdot\sqrt{1+\left(\frac{dy}{dx}\right)^2};$$

or, by (9),

$$\alpha_t = v\frac{\sqrt{g}\varphi'''(x)}{2[-\varphi''(x)]^{3/2}}.$$

Now (9) and (10) can be written in the form $v = F_1(x)$ and $\alpha_t = F_2(x)$, where F_1 and F_2 are power series. The method of inversion of series gives

(11) $$F_2^{-1}(\alpha_t) = F_1^{-1}(v).$$

This entire discussion of the motion of a projectile is valid if the projectile is a finite sphere and also if the medium is not homogeneous. But the relation of α_t to v depends on the character and condition of the medium. So, (11) has the nature of a law for the particular projectile if, and only if, the medium in which the flight occurred was equally resistant at all points in the observed path.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

REPLIES.

34. [*1917*, 134, 341; *1920*, 114, 301.] Given the mixed integral and functional equation

$$\int_{x=0}^{x=h} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

REPLY BY D. C. GILLESPIE, Cornell University.

The formula is a case of Simpson's Rule, where only two intervals are used. If $f(x)$ is a polynomial of degree not greater than three, this approximating formula overplays its rôle and evaluates the integral exactly. The function $f(x)$ being assumed continuous, one might surmise that the formula would not evaluate the integral exactly, for all values of h, unless $f(x)$ were a polynomial of degree three or less. The correctness of this surmise has not been demonstrated. In this note it is shown that *if $f(x)$ is continuous, $0 \leqq x \leqq b$, and has six continuous derivatives, $0 \leqq x \leqq H < b$, and the formula evaluates the integral for each value of h, $0 \leqq h \leqq b$, then $f(x)$ is a polynomial of degree not higher than three, $0 \leqq x \leqq b$.*

Suppose first that a function $g(x)$, continuous in $(0, b)$, satisfies the functional equation

$$\int_0^h x^n g(x)dx = \frac{h}{6}\left\{4\left(\frac{h}{2}\right)^n g\left(\frac{h}{2}\right) + h^n g(h)\right\} \tag{1}$$

for $0 \leqq h \leqq b$ and n a positive integer. The mean value theorem yields

$$\frac{h^{n+1}}{n+1} g(\theta h) = h^{n+1}\left\{\frac{1}{3}\cdot\frac{1}{2^{n-1}} g\left(\frac{h}{2}\right) + \frac{g(h)}{6}\right\}, \qquad 0 < \theta < 1. \tag{2}$$

The function $g(x)$ is continuous; hence, on dividing through by h^{n+1} and taking the limit of both sides as h approaches zero, we obtain

$$\frac{g(0)}{n+1} = \frac{1}{3}\cdot\frac{1}{2^{n-1}} g(0) + \frac{1}{6} g(0).$$

If $g(0) \neq 0$ we have

$$\frac{1}{n+1} = \frac{1}{3}\cdot\frac{1}{2^{n-1}} + \frac{1}{6}.$$

This equation is satisfied for n equal to 1 or 2 or 3 and for no other positive integral value of n. Hence, if n is to be greater than three, $g(0)$ must be zero.

We assume next $g(0) = 0$ and again find an upper limit for n. Let $\bar{h}$ be a value of x for which $|g(x)|$ assumes its upper limit in $(0, b)$. Equation (2) may now be written

$$\frac{\bar{h}^{n+1}}{n+1} g(\theta\bar{h}) = \bar{h}^{n+1}\left\{\frac{1}{3}\cdot\frac{1}{2^{n-1}} g\left(\frac{\bar{h}}{2}\right) + \frac{g(\bar{h})}{6}\right\}, \qquad 0 < \theta < 1. \tag{2'}$$

or

$$\frac{g(\theta\bar{h})}{n+1} = \frac{1}{3}\cdot\frac{1}{2^{n-1}} g\left(\frac{\bar{h}}{2}\right) + \frac{g(\bar{h})}{6}.$$

But

$$\left|g\left(\frac{\bar{h}}{2}\right)\right| \leqq |g(\bar{h})| \quad \text{and} \quad |g(\theta\bar{h})| \leqq |g(\bar{h})|$$

hence

$$\left|\frac{g(\bar{h})}{n+1}\right| \geqq \left|\frac{1}{3}\cdot\frac{1}{2^{n-1}} g\left(\frac{\bar{h}}{2}\right) + \frac{g(\bar{h})}{6}\right| \geqq \left(\frac{1}{6} - \frac{1}{3}\cdot\frac{1}{2^{n-1}}\right)|g(\bar{h})| \text{ for } n \geqq 2.$$

Now

$$\frac{1}{n+1} < \frac{1}{6} - \frac{1}{3} \cdot \frac{1}{2^{n-1}} \text{ for } n \geqq 6.$$

Equation (2′) then can not be satisfied for $n \geqq 6$, unless, of course, $g(x)$ is identically zero.

Suppose now that a function $f(x)$ which satisfies the given equation

$$\int_0^h f(x)dx = \frac{h}{6}\left\{ f(0) + 4f\left(\frac{h}{2}\right) + f(h) \right\}, \qquad 0 \leqq h \leqq b \tag{3}$$

is continuous in $(0, b)$ and has six continuous derivatives in an interval $(0, H)$ in $(0, b)$. The Taylor expansion theorem applied for values of x in $(0, H)$ gives

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{\text{iv}}(\theta x)\frac{x^4}{4!}, \qquad 0 < \theta < 1.$$

The quantity θ is, to be sure, not a constant but a function of x; nevertheless $f^{\text{iv}}(\theta x)$ is a continuous function of x in $(0, H)$. Susbtituting in equation (3) for $f(x)$ this expansion we obtain

$$\int_0^h \frac{x^4}{4!} f^{\text{iv}}(\theta x)dx = \frac{h}{6}\left\{ 4\left(\frac{h}{2}\right)^4 \frac{f(\theta h/2)}{4!} + \frac{h^4 f(\theta h)}{4!} \right\}.$$

This equation, as we have already seen, requires that $f^{\text{iv}}(0) = 0$; and one shows in the same way that $f^{\text{v}}(0) = 0$. Taking the sixth degree term in the expansion,

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{\text{vi}}(\theta x)\frac{x^6}{6!}, \qquad 0 < \theta < 1.$$

If this expansion for $f(x)$ is substituted in (3) there results

$$\int_0^h \frac{x^6 f^{\text{vi}}(\theta x)}{6!}\,dx = \frac{h}{6}\left\{ 4\frac{h^6 f^{\text{vi}}(\theta h/2)}{2^6 \cdot 6!} + \frac{h^6 f^{\text{vi}}(\theta h)}{6!} \right\}.$$

We have shown that this equation can hold only when $f^{\text{vi}}(\theta x)$ is zero for all values of x in $(0, H)$. In the interval $(0, H)$ then $f(x)$ coincides with a cubic polynomial.

In QUESTIONS AND DISCUSSIONS *1920*, 302, it was pointed out by the EDITOR that two continuous functions satisfying equation (3) and coinciding in some interval from the origin to the right coincide throughout. Thus the result announced at the beginning is established.

The theorem, just quoted and used in the proof, is capable of a slight extension; *i.e.*, if a continuous function $f(x)$ satisfies equation (3) and is zero for $a \leqq x \leqq c$ where $c > 2a > 0$, then it is zero throughout $(0, b)$. For suppose $c \geqq h \geqq 2a$, then

$$\int_0^h f(x)dx = \frac{h}{6} f(0).$$

Since, however, $f(x) = 0$, $a \leqq x \leqq c$, as h increases from $2a$ to c the integral remains constant, hence $f(0) = 0$. Thus for $a \leqq h \leqq c$,

$$\int_0^h f(x)dx = \frac{2}{3} h f\left(\frac{h}{2}\right),$$

and as the integral maintains from $h = a$ to $h = c$ a constant value, which we have seen must be zero,

$$f\left(\frac{h}{2}\right) = 0, \qquad a \leqq h \leqq c,$$

or

$$f(x) = 0, \qquad \frac{a}{2} \leqq x \leqq \frac{c}{2}.$$

Continuing this process, we see that $f(x) = 0$, $0 \leqq x \leqq c$. Then by the previously quoted remark of the EDITOR, $f(x) = 0$ in the entire interval $(0, b)$.

DISCUSSIONS.

As the first discussion this month we present a paper which was read at the last annual meeting of the Mathematical Association of America, as part of a program devoted to the consideration of the sort of training in mathematics most useful for students specializing in fields in which mathematics finds frequent application. Professor Reed represents the point of view of the biometrist. His recommendations, briefly summarized, are: the usual courses in algebra, trigonometry, and analytic geometry; a short course in the calculus, emphasizing principles rather than technique; and a course in probability, with stress on statistical theory and the adjustment of curves to given data. It would seem, therefore, that the student of biometry will generally find ready at hand, in most of our colleges, courses agreeing reasonably well with the plan outlined by Professor Reed. Probably, too, the textbooks in elementary subjects are no longer so restricted in their treatment as Professor Reed implies. Few, if any, trigonometries of recent date, for example, fail to give applications to mechanics.

The brevity of the course in calculus in Professor Reed's scheme should be considered rather carefully. Is a student of statistics to accept Stirling's formula and the value of the probability integral on faith, or is he to receive demonstrations? The proofs, if given with logical precision, require more thorough treatment of the behavior of series and improper integrals than is usually found in even rather extensive first courses in calculus.

In the second discussion, Mr. Webb outlines a treatment of complex numbers, including the definition of exponential and trigonometric functions and their relationship. His scheme is similar to that found in some textbooks in trigonometry; seldom, if ever, is so full a discussion included in the ordinary course in algebra. Two items in his outline call for special comment. No. 6 prescribes "the customary development of e and of e^x by the binomial theorem as the limits of $(1 + 1/n)^n$ and $(1 + 1/n)^{nx}$ as $n \doteq \infty$." This customary scheme involves either a scandalously inaccurate treatment of the double limit process (as in most textbooks on calculus) or a degree of logical precision scarcely within the reach even of the average college graduate. The question involved is much more delicate than that of mere convergence. Either the properties of uniform convergence, or else some equivalent special process to avoid this concept must be used. It would be desirable, if possible, to find some less difficult approach to the exponential function. No. 8 implies that there must exist some k such that $\cos 1 + i \sin 1 = e^k$. Since the series for $\cos \theta$ and $\sin \theta$ are determined on the hypothesis that such a k exists, and then by use of the series the value $k = i$ is obtained, it is not easy to see how to avoid the hypothesis.

Professor Bell points out that the method of proof known as mathematical induction is valid only by virtue of a distinct assumption, which he formulates very clearly thus: *If a theorem is true for $n = 1$, and if its truth for $n - 1$ implies that it is true for n, then the theorem is true for all whole numbers.* This discussion should be helpful to our readers, inasmuch as the necessity of this explicit assumption is not always clearly recognized by teachers. However, the author's

implication that the rôle of such an assumption is overlooked by specialists in the logic of mathematics seems to be unwarranted. Writers on the foundations of mathematics generally recognize that the validity of mathematical induction is the result of a pertinent hypothesis. A common alternative to Professor Bell's form of statement is the following: *In any set of positive integers there is a least integer*. From this hypothesis, the other form of statement can be proved by considering the set of all numbers n (if any) for which the theorem under consideration is false.

Poincaré himself, in the sections of *Science and Hypothesis* criticized by Professor Bell, states that the principle of mathematical induction can not be obtained either from syllogisms based on the other fundamental axioms or from experience. For example[1]: "We may readily pass from one enunciation to another, and thus give ourselves the illusion of having proved that reasoning by recurrence is legitimate. But we shall always be brought to a full stop—we shall always come to an indemonstrable axiom, which will at bottom be but the proposition we had to prove translated into another language." Russell has long regarded mathematical induction as a *definition* of the *natural numbers*:[2] "We *define* the natural numbers as those to which proofs by mathematical induction can be applied." Since the number system of elementary arithmetic and algebra contains no other *integers* except these "natural numbers," it is in this special case only a matter of terminology to say which of the fundamental hypotheses shall be called *definitions* rather than merely *axioms*. Many logicians would prefer to regard the whole set of fundamental hypotheses underlying any mathematical system as collectively *defining* the system.

Professor DeCou gives an instance of a problem in maxima and minima arising in the printing office. The solution is simple; and in fact it may be shown by a slight alteration in notation that the problem belongs to the familiar type in which a sum of two variables is to be minimized while the product remains constant. It is published here on account of the value for the class room of problems arising from practical sources. How many similar instances may occur in trade, industry, and every day life, in which a little knowledge of mathematics would save, as here, "an added cost of $100 to $200," or an equivalent in energy or convenience?

One detail of the problem suggests a further interesting question. As the calculus refuses to discriminate between integers, fractions, and irrationals, the formal solution of the problem generally gives an irrational result for the number of electrotypes needed. Professor DeCou says, "Of course only the nearest integral value of x is used." It is fairly clear, since the relationship of C and x is represented graphically by a hyperbola, that the minimum C for integral x must be given by one of the two integers enclosing the value of x for which the actual minimum occurs. But it is not evident that of these two, the *nearest* will give the value desired. In fact, this turns out not to be precisely correct,

[1] *Science and Hypothesis*, Part I, Chapter VI.
[2] *Introduction to Mathematical Philosophy*, Chapter III.

and our readers may be interested in verifying the true result. If $\sqrt{PR/ES} = \alpha$, then α should be replaced by the next smaller integer n or the next greater integer $n + 1$, not according as α is less or greater than the arithmetic mean $n + \frac{1}{2}$, but according as α is less or greater than the geometric mean $\sqrt{n(n+1)}$. For large values of n, the distinction between the two criteria is slight. Of course in any actual case it would be simple enough to substitute each of the two values in the expression for C in order to determine which was the better.

I. The Mathematics of Biometry.

By Lowell J. Reed, Johns Hopkins University.

(Read before the Mathematical Association of America, January 1, 1920.)

I think we are all agreed that the program for this meeting of the Mathematical Association of America is as important as any that the Association has ever discussed. Any science that fails to ally itself with the other sciences must of necessity have a narrow development, and this is perhaps more true of mathematics, due to its wide application, than of any of the other sciences. It is to be hoped that the Program Committee will carry out its own suggestion that at some later meeting we consider the converse of the present question, that is, the contribution of other sciences to the development of mathematics.

My own part in the present program is the discussion of the mathematics of biometry, and instead of presenting illustrations of the application of mathematics to this branch of biology I am going to outline what I consider to be the proper mathematical training for work in biometry. I wish first to call attention to the development of the present method of teaching mathematics. We have in the field of mathematics a number of branches that have been developed mainly as tools for the solution of practical problems. Thus, trigonometry has been developed mainly for the solution of problems in surveying; probability has been developed for games of chance. Now whenever this has been the case the teaching of the subject has been concentrated on this one particular application, to the exclusion of all others. In recent years the increasing use of mathematical methods in such sciences as chemistry, biology, etc., has led to an effort to broaden the teaching of mathematics by introducing new applications. The result has been a new group of textbooks under such titles as "Calculus for Chemical Students," "Mathematics for Agricultural Students," etc. All of these texts seem to me to miss the point in that they imply that calculus for chemistry students is distinct from calculus for other groups. This I do not believe to be the case. If we consider the mathematical needs of a student in any one of the sciences we find them about as follows: First, he needs a foundation in algebra, trigonometry, analytic geometry, and calculus; and secondly, he needs to be trained to take a problem in his particular field and translate it into mathematical language. The latter need is the greater of the two, and it is the one that it is the more difficult to satisfy.

I now wish to present what seems to me to be the proper mathematical training for work in biometry; although the following courses are outlined with the needs of a student of biology in mind, I feel sure that they are the proper ones for students in any branch of science. The first year courses in mathematics should cover the three subjects, algebra, trigonometry, and analytic geometry. The course in algebra should be fully as complete as the one at present given and may occupy the first half of the year. Emphasis should be laid on problem solution with the idea of satisfying the second of the needs listed above. The course in trigonometry should give to the students a knowledge of the trigonometric functions, showing their relations to each other and to the right-angled triangle; it should develop the relations that exist between the function of the angle and the sides of the general triangle, but it should not contain the usual extensive drill in the application of these formulas to the solution of triangles. The use of the Law of Sines and Law of Cosines in mechanics is fully as important as their use in solving triangles and the application is not obscured by a long computation, but one very seldom sees such applications in a textbook on trigonometry. The course in analytic geometry should give the methods of the subject with a fairly complete discussion of the straight line and circle, and a brief discussion of conics. In this course again there is an excellent opportunity to train students in the methods of expressing ideas in mathematical notation. To the first year's work might well be added a short section on computation so that the students could become acquainted with the use of logarithms, and if possible learn to use some of the various calculating machines.

The second year's work should be divided into two parts. The first half year should be devoted to a course giving the elements of calculus, emphasizing principles and methods rather than the tricks of differentiation and integration. In the second half of the year the students should be divided into groups according to their interests, one group for those majoring in engineering, another for those majoring in the natural sciences, etc., the divisions formed being adjusted to suit the needs of the institution in question. Then the students in each group should be given work applying the preceding mathematics course to their respective fields. The students majoring in different natural sciences should not in my opinion be further separated into distinct groups as they are all concerned with the same fundamental problem, that is, the analysis of observed data, and the course given to this group of students should train them in modern statistical methods of analysis. First, the subject of probability should be presented from its theoretical side. This should include the fundamental definitions and a discussion of probability theory leading to the normal curve. The meaning of the constants of the normal curve and the notion of probable error should be introduced at this point. The subject of probability should be followed by a discussion of the principle of moments, illustrations being drawn from mechanics. The treatment of this subject should lead naturally into a discussion of frequency distributions and correlation. The relationship of the first, second, and third moments to the mean, standard deviation, and the skewness of frequency distribution, should be

clearly brought out, mechanical illustrations being freely used. The correlation coefficient should be developed as a product moment with the mean of the variables taken as an origin. Following this the students should be taught the theory of curve fitting by the method of moments and this theory should be illustrated by examples in the derivation of empirical formulas. The method of fitting curves by least squares need not be presented as it is not so widely applicable as the method of moments. The student should also have impressed upon him the importance of the idea of probable error and should be trained to work out the probable error of all such constants as mean, standard deviation, coefficient of correlation, etc.

The course just outlined would furnish a student in biometry with the proper mathematical foundation for his work; his mathematical needs are similar to those of the student of any natural science. One of the most common quotations among biometricians is that of Sir Francis Galton: "Until the phenomena of any branch of knowledge have been submitted to measurement and number it cannot assume the status and dignity of a science," and as we see more and more of the leading scientists taking this point of view it impresses upon us the growing importance of mathematics. The methods of teaching mathematics should develop accordingly; and if this present meeting succeeds in stimulating the teaching of mathematics to live up to its increased importance, it will indeed have been a success.

II. Complex Numbers in Advanced Algebra.

By H. E. Webb, Central High School, Newark, N. J.

In many electrical laboratories it is desired that the workers have a certain familiarity with vectors, and in particular with the exponential notation $e^{i\theta}$. Examination of texts in algebra most commonly used in schools fails to bring to light an elementary explanation of this notation. Certain texts in trigonometry give such an explanation, but usually in a rather intricate form, and without a clear statement as to what it is all about. The following brief outline is suggested as an addition to a course in advanced algebra for the fourth year in high school or the freshman year in college; in this it is desired to escape from exhaustive analysis of various points which seemingly of necessity must be brought into any *rationale* of this notation.

1. The rules of combination of complex numbers to be presented as an arithmetic of number pairs, *without reference to Argand's diagram.*

2. The sine and cosine of *real positive and negative numbers* to be defined by reference to a unit circle, without mention of other trigonometric functions, leading only to the facts that in a circle of radius r the corresponding lines are $r \sin \theta$ and $r \cos \theta$, and that $\sin^2 \theta + \cos^2 \theta = 1$ for all values of θ.

3. The development of formulas for $\sin (x \pm y)$ and $\cos (x \pm y)$.

4. The polar notation of a complex number, shown as follows: If $a + ib$ is

written $\sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}+i\frac{b}{\sqrt{a^2+b^2}}\right)$, the real and imaginary coefficients may be regarded as the sine and cosine of some number, since the sum of their squares is unity. For the present the number is shown geometrically as an arc of a unit circle, the real coefficient being chosen arbitrarily as a cosine. Hence $a+ib$ may be written $r(\cos\theta+i\sin\theta)$.

5. DeMoivre's theorem, as usually presented, attention being called to the fact that in the case of $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$, the argument seems to possess the characteristics of a logarithm, where n is any rational number.

6. The customary development of e and of e^x by the binomial theorem, as the limits of $(1+1/n)^n$ and $(1+1/n)^{nx}$ as $n\doteq\infty$. It is necessary at this point to make reference, of course, to convergency of series. It is not necessary to exhaust this topic, for the simple reason that the exponential series

$$e^x=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}\cdots$$

looks to be convergent, since there is always a term beyond which the successive multipliers of the denominator exceed the corresponding multipliers of the numerator, for any value of x.

7. Since this series holds for all rational values of x, it will serve as a definition of the *meaning* in the case of an irrational exponent, and, as well, of an imaginary exponent, and also of a complex exponent, if $e^a\cdot e^{ib}$ is written e^{a+ib}. In the latter case $(a+ib)^n$ would require to be developed by the binomial theorem. It is *assumed* explicitly, though not proved, that the laws of rational exponents extend to complex exponents of powers of e.

8. Euler's theorem, established as follows:

Let $\cos 1+i\sin 1=e^k$, where k is a complex number to be determined.

Then $\cos 1-i\sin 1=e^{-k}$, since the product of these two equations, member for member, gives identity.

By De Moivre's theorem

$$\cos\theta+i\sin\theta=e^{k\theta},$$
$$\cos\theta-i\sin\theta=e^{-k\theta},$$

whence

$$\cos\theta=\frac{e^{k\theta}+e^{-k\theta}}{2},\qquad i\sin\theta=\frac{e^{k\theta}-e^{-k\theta}}{2}.$$

Now if $e^{k\theta}$ and $e^{-k\theta}$ are developed into series by the definition of e^x, and the series added and halved, term for term, it follows that

$$\cos\theta=1+\frac{k^2\theta^2}{\lfloor 2}+\frac{k^4\theta^4}{\lfloor 4}\cdots,$$

while

$$i\sin\theta=k\theta+\frac{k^3\theta^3}{\lfloor 3}+\frac{k^5\theta^5}{\lfloor 5}\cdots.$$

Dividing both members of the last equation by θ,

$$i\left(\frac{\sin\theta}{\theta}\right) = k + \frac{k^3\theta^2}{\lfloor 3} + \frac{k^5\theta^4}{\lfloor 5} \cdots.$$

As $\theta \doteq 0$, $(\sin\theta)/\theta \doteq 1$, and since k is constant,

$$k = i.$$

Substitution of i for k in the various formulas cited affords series development of $\cos\theta$ and $i\sin\theta$, in the first of which all terms are real, and in the second all imaginary; and also establishes the notation

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

9. If at this stage *and not sooner* the vector is defined the student will grasp perfectly the idea that such a directed line has two values, an algebraic value and an absolute value, which are different, in that its algebraic value depends upon its direction determined by the length of the arc of a unit circle described about the origin as a center.

It is the experience of the writer that in secondary school classes this development requires about as many days as he has topics outlined above, and that certain difficulties relating to the *value* of the ordinate are thereby avoided. A favorable opening is also provided for the teaching of hyperbolic functions, when this is desirable.

III. On Proofs by Mathematical Induction.

By E. T. Bell, University of Washington.

1. The best way to cure oneself of a crotchet is to confide it to some sympathetic listener. The crotchet in this note is one which has worried me since school days when I was induced to repeat the proof of the binomial theorem by mathematical induction. The same crotchet seems to trouble successive generations of freshmen, for occasionally one has obstinacy enough to balk at the magic formula "*and therefore the theorem is always true,*" with which many authors conclude their proofs by recurrence. I hold that mathematical induction has no place in elementary teaching, particularly when such teaching strains at mathematical gnats, as in pseudo-rigorous presentations of the elementary theory of limits, the better to swallow logical camels such as some proofs of the binomial theorem or their equivalent quoted presently from Poincaré. This is the crotchet. In short, elementary teaching would be more convincing if it left rigor to that logistics which was Poincaré's bête noir.

2. It would be difficult to find a balder statement of the logical vice which characterizes many proofs by mathematical induction in the current text books, than the following extract from Poincaré's essay *On the Nature of Mathematical Reasoning,* in *Science and Hypothesis* (Halsted's translation, page 36, section IV).

Having considered three examples of "proof by recurrence," and having drawn a false conclusion in each of them, Poincaré says:

"Here I stop this monotonous series of reasonings. But this very monotony has the better brought out the procedure which is uniform and is met again at each step.

"This procedure is the demonstration by recurrence. We first establish a theorem for $n = 1$; then we show that if it is true of $n - 1$, it is true of n, and thence conclude that it is true for all whole numbers."

It is only fair to state that in the next section (V) of his essay, Poincaré gives an unobjectionable form of "this procedure," which he calls "mathematical reasoning *par excellence*." This, however, may have been an inadvertence, as the closing paragraph of the essay again exhibits the circularity of the reasoning in all its viciousness:

"Observe finally that this induction is possible only if the same operation can be repeated indefinitely."

From this we should expect a rich crop of subtle fallacies when "this induction" is applied to prove that a certain assemblage contains an infinity of members, or when it is used to demonstrate the universal truth of a proposition.

3. To exhibit the logical defect in this "mathematical reasoning par excellence," let us separate Poincaré's summary into its three constituents:

(1) "We first establish a theorem for $n = 1$;

(2) "Then we show that if it is true of $n - 1$, it is true of n;

(3) "And thence (we) conclude that it is true for all whole numbers."

In (3) "thence," if it means anything definite, must refer to (2) and (1). That is, the conclusion that the theorem is true for *all* whole numbers is to follow from (1) and (2) *only*. It does not so follow. In order to draw the conclusion (3), we need, either as a postulate, or, if it can be proved from simpler assumptions, the proposition:

(4) If a theorem is true for $n = 1$, and if its truth for $n - 1$ implies that it is true for n, then the theorem is true for all whole numbers.

Without (4), all that (1) and (2) give is the means for *step by step* assertions that the theorem is true in successive cases. Thus, supposing (1) and (2) established, if it be required to see whether the theorem is true for $n = 5$, we must, for all that (1) and (2) prove, take the steps 1 to 2, 2 to 3, 3 to 4, 4 to 5, omitting none. Or, again, if (1) and (2) are established, and we wish to know whether the theorem is true for $n = 9^{9^9}$, say, we must take $9^{9^9} - 1$ steps in order to find out, for neither (1) nor (2) permits us to take more than one step at a time. In this case we might never gratify our curiosity, and the truth or falsity of the theorem for $n = 9^{9^9}$ would remain as inaccessible to our knowledge as is the other side of the moon. Nor could we assert that the theorem *most probably* is true for this value of n, for we cannot predicate probabilities in the absence of data. Our belief that the theorem is true for this value of n might be strong; but belief belongs to the realm of emotional experiences, and is seldom conspicuous for the reasonableness of its tenets. Until (4) or its equivalent is proved or admitted

as a postulate, it would seem to be advisable to banish such phrases as "all whole numbers," and "the same operation can be repeated indefinitely," from elementary texts which make pretensions to rigor.

4. If reasoning by recurrence is, as Poincaré claims, mathematical reasoning *par excellence*, and if the objections put forth in § 3 are not groundless, it would seem to follow that mathematics is like any other science in that the conclusions which it legitimately draws are no more "general" or "universal" than those of other sciences. This contradicts what seems to be a current valuation of mathematical truth in the minds of laymen and some others who hold that mathematics has a timeless, eternal aspect, independent of all the empiricism which characterizes the conclusions of physical sciences.

There is one way of escape which is so obvious that it need only be pointed out. We can beat the mathematical devil round the logical bush by saying that (4) of § 3 is the rule, or law, of inference. But it would be a wise logician indeed who recognized (4) as one of his legitimate children. For where is either a proof of it or its explicit statement as a postulate of logic to be found?

IV. A Practical Printer's Problem in Maxima and Minima.

By Edgar E. DeCou, University of Oregon.

Dean Eric W. Allen, of the School of Journalism of the University of Oregon, presents a very interesting problem of frequent occurrence to the practical printer. The printer's only method of solution is by trial and error; and he states that on a large job of printing an added cost of $100 or $200 is often incurred by inability to solve the problem.

The conditions of the problem are as follows: 200,000 (P) prints are required; 1200 (S) prints per hour is the speed of the press; $2.00 ($R$) per hour is the cost of running the press; 55 cents (E) each is the cost of the extra electrotypes, needed after the type is once set up. Required the number of electrotypes (x) that should be used to secure the minimum cost (C).

The problem is evidently one in determining the minimum value of C by the use of the differential calculus. The particular case takes the form,

$$C \text{ (in cents)} = \frac{200{,}000 \times 200}{1200(1+x)} + 55x = \frac{100{,}000}{3(x+1)} + 55x,$$

where x represents the number of electrotypes. Differentiating

$$\frac{dC}{dx} = -\frac{100{,}000}{3} \cdot \frac{1}{(x+1)^2} + 55 = 0,$$

for minimum value of C. Hence

$$x = \frac{100}{33} \cdot \sqrt{66} - 1 = 23.6+.$$

In other words, the most economical number of electrotypes to use for this job is 24. Of course only the nearest integral value of x is used.

The general problem is stated thus:

$$C = \frac{PR}{S(1+x)} + E \cdot x,$$

$$\frac{dC}{dx} = -\frac{PR}{S(1+x)^2} + E = 0,$$

for a minimum.

From which

$$x = \sqrt{\frac{PR}{ES}} - 1.$$

This gives a formula involving only the arithmetical work of finding the square root to determine the number of electrotypes needed in any given case, and one of easy application by any practical printer.

RECENT PUBLICATIONS.

REVIEWS.

Differential Equations. By H. BATEMAN. London, Longmans, Green and Co., 1918. 8vo. 11 + 306 pp. Price 16 shillings.

The study of elementary methods of integrating differential equations is one which is taken up in many American colleges in a course following the integral calculus, or sometimes as a part of that course. When properly taught, it is a subject admirably adapted to developing in the student a skillful technique in using his calculus, a thing which he will find most helpful in his later work. Many students coming from calculus are woefully weak in many parts of the work which they have studied and "passed," so if such students are to go on to differential equations the beginning, at least, must be easy. They will then have some chance to develop and show their real ability. However, the manipulative side of the study must not be over-emphasized, for the extensive theoretical parts must be suitably developed. Moreover, there is far more opportunity for geometrical discussions than is generally given.

Since the time of Boole many text books on elementary differential equations have appeared in England and America. The general plan of all these books has, however, been much the same. Differential equations were classified into certain "standard forms," and, after having discussed the methods to be used in integrating these type forms, problems were given falling more or less closely under them. The number of real "clothed problems" was usually small. The book here under review is entirely different both in arrangement and content from

the earlier books. This is especially true in the first four chapters. To quote from the preface: "The order in which the material has been arranged is slightly different from that which is usually adopted. Instead of beginning with the standard forms of equations which can be solved very easily, I have devoted the second chapter to integrating factors, and the third to the method of transformations."

When one comes to examine those changes in arrangement they appear to be considerable in the early chapters. The book has eleven chapters as follows: I. Differential equations and their solutions. II. Integrating factors. III. Transformations. IV. Geometrical applications. V. Differential equations, with particular solutions of a specified type. VI. Partial differential equations. VII. Total differential equations. VIII. Partial differential equations of the second order. IX. Integration in series. X. The solution of linear differential equations by means of definite integrals. XI. The mechanical integration of differential equations. At the end there are eight pages of miscellaneous examples, and an index.

With this list of chapter headings before him, let the reader ask himself where he would look for some topic with which he is familiar. Take, for instance, the general linear equation with constant coefficients. The index does not help, but a process of elimination leads one to look in the table of contents under Chapters II and III. Search there fails to give an exact reference, but linear second order equations are treated on p. 28 of Chapter II. Eight pages later we find the general linear equation with constant coefficients. However, the author does not treat it by means of an integrating factor, but by the symbolic method.

The question of the arrangement of the subject matter is of course extremely important. If a new arrangement is better than the old one, such comments as the foregoing would mean little. The author's plan in the first four chapters seems to be to classify his equations according to the methods used in discussing them, but he abandons this plan in the later chapters. The older plan classified differential equations according to their form, and integrated them by any available method. This older plan has one great advantage over the author's method, for it gives the student a classification of the equations themselves. When an ordinary differential equation is proposed for solution, the plan of attack is to discover which one of many methods is to be used. For this purpose some method of classification must be followed, and the most obvious classification is first according to order, and then by the form of the equation. If then the text book follows this same classification the student will be more ready to use it in attacking an unsolved equation. On the other hand the author's plan has the obvious advantage arising from the application of somewhat similar methods to equations of widely different forms. It leads to some curious results. Thus we find discontinuous solutions of linear equations with constant coefficients and a discussion of the general theory of a simultaneous system of n linear equations coming before such simple matters as the integrating factor of an equation of the

first order and degree, and the solution of the homogeneous equation of the first order. Such reversals of difficulty are frequent throughout the book. If Chapters II and III were interchanged they would be easier for the student. However, the author has small regard for the student's ease. Formidable computations occur frequently. Perhaps this conduces to that manipulative skill which the reviewer advocated in the beginning.

The book contains a large amount of valuable material, both in the text and in the problems. In the text, under ordinary differential equations, the discussion of linear equations and simultaneous systems of linear equations is especially complete. The author makes many interesting applications of these results to the discussion of problems in electricity, radioactivity, and dynamical systems. Many of these involve far more extensive investigations of the supplementary conditions than are given in the older texts, and some of them involve discontinuous solutions. Among the problems for the student to solve are many more of the same kind.

In Chapter V we find discussions of Euler's equation, the hypergeometric equation, Bessel's and Legendre's equations. These equations are further treated in Chapters IX and X. In Chapter IX. are also given Cauchy's existence theory for the ordinary first order equation, the method of successive approximations for that equation and Runge's method of approximate solution; certain convergence proofs for the linear equation of the second order, and for a simultaneous system of two first order equations. Here also we find a discussion of some of the expansion problems of physical mathematics.

In Chapter X we find a satisfactory account of the method of solving linear differential equations by means of definite integrals, and the Green's function of a linear partial differential equation with assigned boundary conditions.

Let us now return to Chapter VI on Partial Differential Equations of the first order. The whole treatment of this chapter is unsatisfactory. It strikes the reader as being one mass of formulas (*e.g.*, pp. 132, 133). That this is unnecessary is seen by consulting Goursat.[1] The author leaves almost untouched the whole of the beautiful geometrical relations between the various integral surfaces and the characteristic strips, as developed, for example, by Lie.[2] It is true that the author derives the differential equations which define the characteristic strips, but he scarcely uses the result. The fact that the integral surface is in general determined by making it pass through a given curve, but fails to be determined when and only when that curve is a *characteristic* curve, is unnoticed.

Chapter VII deals with total differential equations. In transforming variables in such an equation the author uses a symbolic method and introduces some results of his own.

Chapter VIII opens with a discussion of the homogeneous linear partial differential equation of the second order. This is followed by extensive applica-

[1] *Equations aux dérivées partielles du premier ordre*, Paris, 1891, pp. 90–92 and pp. 102–107.
[2] *Berührungstransformationen*, Leipzig, 1896.

tions to important physical problems, including wave propagation, Maxwell's equations, theory of electrons, Laplace's equation and harmonic equations.

The last chapter describes various mechanical contrivances for solving certain differential equations of special forms. Several of these are due to E. Pascal, who has devised a number of instruments for this purpose.

Throughout the book there are many typographical errors. Most of them are easily corrected by the reader, but at least one of them has caused trouble to an unwary instructor.

The reviewer feels that the book will be difficult reading for a student beginning the study of differential equations, but that can be determined only by trying it out with a class. The book will surely prove to be a valuable addition to the library of the worker in mathematical physics.

CHARLES L. BOUTON.

HARVARD UNIVERSITY,
June, 1920.

The Theory of Determinants in the Historical Order of Development. By THOMAS MUIR. Volume 3, the period 1861 to 1880. London, Macmillan, 1920. 8vo. 26 + 503 pp. Price 35 shillings.

The first edition of the first volume of this work was reprinted in book form, in 1890, from the *Proceedings of the Royal Society of Edinburgh*; a second edition, with over 200 pages of additional material, appeared in 1906, and covered the history of general and special determinants up to 1841. The second volume (1911) made a similar survey for the period 1841 to 1860. The third volume under review covers an additional twenty year period. In June, 1918, the manuscript of a fourth volume bringing the record up to the end of the nineteenth century was nearly complete. Mathematicians must ever be grateful to Sir Thomas for his monumental work which is designed to contain a complete record of published results in connection with the theory of determinants.

The material is admirably arranged and indexed so that it is possible readily to trace the contributions to the theory of any individual, or the chronological development of any special type of determinants. For example chapter 14 in volume 2 and chapter 15 in volume 3 contain the history of circulants from the first paper of Catalan in 1846 to the last of Gegenbaur in 1880; skew determinants may be traced in a similar way in the fourteenth, ninth and tenth chapters of volumes 1, 2 and 3 respectively. Although a three page chapter is devoted to "cubic and n-dimensional determinants up to 1880" and various titles are listed, practically as in the article of 1900 by Professors Hedrick and Cairns,[1] the contents of the papers are not analyzed as in the other chapters because the work in question is a survey of determinants as ordinarily defined, and not of their generalizations.

The titles of the chapters are as follows—I: "Determinants in general, from 1860 to 1880," 1–82; II: "Determinants and linear equations, from 1861 to 1878," 83–93; III: "Axisymmetric

[1] "On three dimensional determinants," *Annals of Mathematics*, second series, vol. 1, pp. 49–67.

determinants, from 1846 to 1879," 94–122; IV: "Symmetric determinants that are not axisymmetric, from 1862 to 1879," 123–131; V: "Alternants from 1860 to 1879," 132–175; VI: "Compound determinants from 1862 to 1880," 176–207; VII: "Recurrents from 1858 to 1879," 208–247; VIII: "Wronskians from 1862 to 1874," 248–256; IX: "Jacobians from 1862 to 1877," 257–271; X: "Skew determinants and Pfaffians from 1862 to 1880," 272–283; XI: "Orthogonants from 1855 to 1879, 284–308; XII: "Persymmetric determinants from 1836 to 1879," 309–326; XIII: "Bigradients from 1859 to 1880," 327–362; XIV: "Hessians, from 1862 to 1879," 363–371; XV: "Circulants, from 1861 to 1880," 372–392; XVI: "Continuants from 1850 to 1880," 393–422; XVII: "Multilineants up to 1877," 423–428; XVIII: "Cubic and n-dimensional determinants up to 1880," 429–431; XIX: "Bordered determinants up to 1880," 432–446; XX: "Determinants whose elements are combinatory numbers up to 1880," 447–462; XXI: "Zero-axial determinants up to 1888," 463–468; XXII: "The less common special forms from 1839 to 1880," 469–496; "List of authors," 497–503.

R. C. ARCHIBALD.

College Teaching. Studies in Methods of Teaching in the College. Edited by PAUL KLAPPER with an Introduction by N. M. Butler. Yonkers-on-Hudson, New York, World Book Co., 1920. 16 + 583 pp.

This book contains twenty-eight chapters by as many different authors. Chapter VIII (pages 161–182) on "The Teaching of Mathematics" is by G. A. MILLER. The sub-headings of the chapter are as follows: Recent changes and some of their sources; Influence of researches in mathematics on methods of teaching; Range of subjects and preparation of students; Variety of college courses in mathematics; History of college mathematics; Relation of mathematics in secondary school and college; Aims of college mathematics: methods of teaching; Advanced work in college mathematics; Mathematics and technical education; Preparation of the college teacher of mathematics; The mathematical text-book.

The Teaching of Arithmetic. By JOHN C. STONE. Chicago, Benj. H. Sanborn & Company, 1918. 262 pp. Price $1.32.

This presents for teachers, supervisors, and those preparing to teach, "a discussion of the aims and purposes of a course in arithmetic and of the methods of presenting each topic that should find a place in our elementary schools." The final chapter, on "Measuring results," gives an account of the standard arithmetic tests developed during the last twenty years.

NOTES.

In T. ZIEHEN, *Lehrbuch der Logik auf positivistischer Grundlage mit Berücksichtigung der Geschichte der Logik*, (Bonn, A. Marucs & E. Webers Verlag, 1920), are discussed: "Die mathematische (symbolistische) Logik" pp. 227–236 including a bibliography of about 35 authors; "Mathematische Grundlegung der Logik," pp. 410–416.

A new list of the members of the Mathematical Association (England) was published in April, 1920. It contains the names of 9 "honorary members," of 747 "ordinary members", and of 83 "associates" of the London, Yorkshire, North Wales, and Sydney (New South Wales) Branches.

Librarians should not be confused by the fact that the last year (October, 1919–September, 1920) of *Revue de Mathématiques Spéciales* is marked "30e année." While the first year of publication was 1890–91, there was no issue between September, 1914 and October, 1919. As the publisher (Vuibert, Paris) acknowledges: "Les 25e, 26e, 27e, 28e, 29e années manquent donc à la collection."

The sixth quarto volume of the sumptuous edition of *Tychonis Brahe Dani Opera Omnia*, now being issued under the auspices of the Danish Society of Language and Literature (Danske Sprog- og Litteratur-Selskab), was published just a year ago. The editor, J. L. E. DREYER, who edited also *The Scientific Papers of Sir William Herschel* (1912), was director of the Observatory at Armagh, Ireland, 1882–1916.

A marble monument to Tartaglia (1499?–1557) was unveiled at Brescia, his birthplace, on November 10, 1918. "L'Ateneo" of Brescia published an "elegante opuscolo, riccamente illustrato," entitled *Scoprendosi il Monumento a Nicolò Tartaglia*, and containing short articles by seven different authors. Among them are: "L. da Vinci e N. Tartaglia" by A. Favaro; "Nicolò Tartaglia e l'arte della guerra" by G. Loria.

The following publications have appeared during the past four months: *Archimedes* (Pioneers of Progress Series), by T. L. HEATH (London, S.P.C.K. Price 2 s.); *Some Famous Problems of the Theory of Numbers and in particular Waring's Problem. An inaugural lecture delivered before the University of Oxford*, by G. H. HARDY (Oxford, Clarendon Press. Price 1 s. 6 d.); *An Introduction to String Figures. A new Amusement for Everybody*, by W. W. ROUSE BALL (Cambridge, W. Heffer and Sons. Price 2 s.); *An Introduction to Combinatory Analysis*, by P. A. MACMAHON (Cambridge University Press. Price 7 s. 6 d.); *Teachers' Manual, Second Course in Algebra*, by W. B. FORD and C. AMMERMAN (New York, Macmillan. 5 + 247 pages. Price $1.20); *Elementary Functions and Applications*, by A. S. GALE and C. W. WATKEYS (New York, Henry Holt and Co. 20 + 436 pages. Price $2.60).

There is much current discussion of "research." The address delivered on November 8, 1919, by Professor J. C. FIELDS, of the University of Toronto, as president of the Royal Canadian Institute, has been recently issued in pamphlet form (27 pages). It is entitled "Universities, research and brain waste."—In the *Scientific Monthly*, July, 1920, Dr. J. R. ANGELL deals with the conception of research, the distribution of research functions, personnel, training research men, organization and cooperation in research, and organization of the National Research Council, in "The organization of research."—A recent address on "The college teacher and research," by Professor LeRoy D. WELD, of Coe College, was published in *Science* for July 16, 1920.—Professor Paul APPELL in an interview (*Temps*, May 22, 1920) defined instruction in the secondary schools and universities: "le premier, c'est là, où *on apprend à apprendre*, le second, c'est celui où *on apprend à chercher*."

The Government Printing Office, Washington, D. C., has only recently distributed to libraries: *Aeronautics, Third Annual Report of the National Advisory Committee for Aeronautics 1917*, which was published in 1918. It is a volume of nearly five hundred royal octavo pages and many plates, and contains twenty-three reports. Report 21, "Theory of an airplane encountering gusts, II" by Professor E. B. Wilson, of Massachusetts Institute of Technology, occupies pages 403–431, and is in continuation of his report published in the first annual report of the Committee.

"In speaking of *time* as a fourth dimension to space, have not our mathematicians been guilty of a carelessness in language which they would be the last to tolerate in symbols? They are always careful to associate (x, y, z) not with t but with ct, where c is the velocity of light. The fourth dimension is not *time* at all, but *distance travelled*, otherwise these same mathematicians would say its dimensions were wrong. Or if it is preferred to use time, then we must divide (x, y, z) by c, which comes to measuring distances in light-years before we can associate them with time.

"A further protest against the rather unnecessary mystification of the layman may not carry quite so much weight; but all the same let it be made. Is it vital to use imaginary time in our language and our thoughts? As a piece of mathematical mechanism it may be useful to treat a hyperbola $x^2 - y^2 = 1$, as a particular case of a circle $x^2 + y^2 = 1$; but it is surely straining our powers of conception needlessly to talk of the hyperbola as a particular case of the circle, except for the purpose of obtaining results quickly by the methods of projective geometry. The 'rigid body rotation' invoked for the transformation of $dx^2 + dy^2 + dz^2 - c^2dt^2$ calls up a false image in some of our minds: it is really a simple distortion with which we have to do, and could not this be mentioned in a cautionary way?"—From *The Observatory*, April, 1920, volume 43, page 171.

ARTICLES IN CURRENT PERIODICALS.

AMERICAN JOURNAL OF SCIENCE, volume 49, May, 1920: "The mathematics of isostasy" I by I. C. Chamberlain, 311–318, II by W. D. MacMillan, 318–323.

JOURNAL OF THE INDIAN MATHEMATICAL SOCIETY, volume 11, no. 6, December, 1919 (published May, 1920): "International Congress of Mathematicians," 201–202; "Computation of the ephemeris of a planet and a comet" by R. N. Apte, 203–218; "An extension of Feuerbach's theorem" by M. Bhimasena Rao, 219–227; "To find a cube the sum of whose divisors is a perfect square" by N. B. Mitra, 228–229; Note on a continued fraction by K. B. Madhava, 230–234; "Astronomical notes," 235–236; Problems and solutions, 237–240.

JOURNAL OF THE WASHINGTON ACADEMY OF SCIENCES, volume 10, no. 2, Jan. 19, 1920: "A trigonometric computer" by F. E. Wright, 29–31.—No. 7, April 4: "A graphical method for plotting reciprocals" by F. E. Wright, 185–188.

MATHEMATICAL GAZETTE, volume 10, May, 1920: "The graphical treatment of differential equations" (continued) by S. Brodetsky, 49–60; "Ratio and proportion" by D. K. Picken, 60–62; "Gleanings from far and near," 62, 74; "Geocentric parallax" by A. A. Krishnaswami Aiyangar, 63–64; "A graphical treatment of simple harmonic motion" by W. G. Bickley, 64–65; "A trigonometrical lucubration" by R. F. Muirhead, 66–68; "Some incidental writings by De Morgan" (continued), 69–74; Review by J. M. Child of W. N. Rose's *Mathematics for Engineers* (London, 1920), 75–76; Review by A. Dakin of C. de la Vallée Poussin's *Leçons sur l'approximation des fonctions d'une variable réelle* (Paris, 1919), 76–77; Review by G. H. Hardy of Petrovitch's *Les spectres numériques* (Paris, 1919), and J. L. S. Hatton's *Theory of the Imaginary in Geometry*, 77–79; "Books received, contents of journals, etc.," i–iv.

MESSENGER OF MATHEMATICS, volume 49, no. 5, September, 1919: "On some series whose nth term involves the number of classes of binary quadratics of determinant $-n$" by L. J. Mordell, 65–72; "A note on a theorem of Riemann's" by Grace C. Young, 73–78; "The twelve elliptic functions related to sixteen doubly periodic functions of the second kind" by E. T. Bell, 78–80.—October: "The twelve elliptic functions related to sixteen doubly periodic functions of the second kind" (concluded) by E. T. Bell, 82–84; "Notes on some points in the integral calculus (lii)" by G. H. Hardy, 85–90; "A new integral equation satisfied by the solution of a certain linear differential equation, which occurs in the theory of electrical oscillations and of the tides" by E. G. C. Poole, 91–96.

NATURE, volume 105, May 13, 1920: "Differential geometry" [review of R. H. Fowler's *The Elementary Differential Geometry of Plane Curves* (Cambridge, 1920)] by G. B. M[athews], 321–322—May 20: "Relativity and Geometry" [review of E. Freundlich's *The Foundations of Einstein's Theory of Gravitation*, translated by H. L. Brose (Cambridge, 1920)] by E. Cunningham, 350–351; "A new method for approximate evaluation of definite integrals between finite limits" by A. F. Dufton, 354–355 ["An approximate evaluation of $\int_0^1 F(x)dx$ is $\cdots\frac{1}{4}\{F(\frac{1}{10}) + F(\frac{4}{10}) + F(\frac{6}{10}) + F(\frac{9}{10})\}$"].—June 3: "A new method for approximate evaluation of definite integrals between finite limits" by C. F. Merchant, 422 ["The subject has a particular interest for naval architects, inasmuch as the majority of calculations relative to displacement, stability, strength, etc., of ships involve the finding of areas and volumes bounded by curved lines and surfaces.

The particular rule enunciated by Mr. A. F. Dufton in *Nature* of May 20 has been in use at this [Royal Naval] College for some years, and gives very accurate results in obtaining areas and volumes, and also, by a further application, the positions of their centers of gravity . . ..

An interesting paper dealing with this subject and giving a great variety of rules for approximate integration was read at the Institution of Naval Architects in 1908 (*Trans. I. N. A.*, Vol. 1) by Sir W. S. Abell entitled "Two notes on ship calculations."]

LA NATURE, volume 48, May 8, 1920: "Un nouvel appareil enregistreur pour l'étude des lois de la dynamique et la composition des mouvements vibratoires" by Paul Bud, 230–234—May 15: "Le centenaire de la machine à calculer industrielle" by L. Reverchon, 249–252; "Il y a quinze ans (27 août 1904) *La Nature* donnait un important et substantiel article de M. Maurice d'Ocagne sur les machines à calculer. L'auteur passait successivement en revue les additionneurs avec ou sans touches, les appareils à multiplier par additions successives et par application du principe des tables de Pythagore, les machines à différences spécialement employées à la construction des tables et les machines algébriques et analytiques dont quelques unes sont extraordinairement compliquées.

On pouvait lire dans cet article les lignes suivantes: 'C'est au financier alsacien Thomas, de Colmar, que revient sans conteste le très grand mérite d'avoir réalisé la première machine à multiplier et à diviser rapide, robuste et fonctionnant en toute sûreté. C'est en 1820 que Thomas créa son Arithmomètre dont depuis lors, le type n'a cessé de se perfectionner sous la direction du constructeur Payen. Très répandu dans les grands établissements financiers, il a fourni une carrière qui a depassé aujourd'hui trois quarts de siècle, attestant de hautes qualités pratiques.'" The article contains a portrait of Thomas, of Blaise Pascal, "inventeur de la premiere machine à calculer," and of Léon Ballée, "a 19 ans, manipulant une machine à calculer."]

NOUVELLES ANNALES DE MATHÉMATIQUES, volume 79, February, 1920: "Sur le maximum et le minimum des fonctions de deux variables" by G. Valiron, 41–50; "Equation angulaire d'un conoïde droit. Application au cylindroide envisagé dans ses rapports avec la distribution des courbures autour d'un point d'une surface" by M. d'Ocagne, 51–55; "Sur les contacts des sphères tangentes à quatre plans" by V. Thébault, 55–59;" A propos de la transformation par tangentes orthogonales" by F. Balitrand, 59–60; "Certificats de calcul différentiel et intégral," 60–73; "Chronique," and "Questions," 76–80—March: "Exposé élémentaire d'une théorie rigoureuse des liaisons finies unilatérales" by E. Delassus, 81–93; "Sur les polygones harmoniques d'un nombre pair de cotés et sur certains cercles du triangle" by V. Thébault, 94–100; "Sur les tangentes aux trajectoires des sommets d'un triangle qui se déforme dans un plan," by R. Goormaghtigh, 100–102; "Chronique," 103–105; "Certificats de calcul différentiel et intégral" 105–117; "Questions," 117–120—April: "Sur l'application de la loi de Gauss à la position probable d'un point dans le plan ou dans l'espace" by J. Haag, 121–142; "Sur un théorème de Cornu relatif aux caustiques" by T. Lemoyne, 142–145; "Chronique," 145–147; "Bibliographie," "Certificat d'astronomie," "Certificat de physique mathématique," "Questions," 147–160.

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY, series 2, Vol. 19, part 1, June, 1920: "Groups involving three and only three operators which are square" by G. A. Miller, 51–56.

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES, volume 6, no. 4, April, 1920: "The starting of a ship" by J. K. Whittemore, 182–185; "A thermodynamic study of electrolytic solutions" by F. L. Hitchcock, 186–197; "Functionals invariant under one-parameter continuous groups of transformations in the space of continuous functions" by I. A. Barnett, 200–204.

REVUE DE L'ENSEIGNEMENT DES SCIENCES, volume 14, January–February, 1920: "Elimination d'une inconnue entre plusieurs équations" by M. Stuyvaert, 1–6; "Apropos de la notion

d'aire vectorielle d'un contour gauche" by G. Bouligand, 6–9; "La transformation apsidale et le problème de Monge" by F. Meyer, 9–14; Sur le volume engendré par un triangle en tournant autour d'un axe" by J. Juhel-Rénoy, 14–17; "Résolution graphique de l'équation $a \cos x + b \sin x = c$" by H. Girard, 17–20; "Ecole Normale Supérieure (1919), concours spécial," "Ecole Polytechnique (1919), concours spécial," etc. 29–48.

Revue de Mathématiques Spéciales, volume 13, no. 6, March, 1920: "Résolution de l'équation du 3e degré" by P. Maurice, 129–131; [solutions of problems in analytic and descriptive geometry], 131–139, 141–149; [Questions proposed in oral examinations at the Ecole Polytechnique, Ecole Centrale, etc.], 139–140, 149–152—No. 7, April: "Resolution de l'équation du 3e degré" (conclusion) by P. Maurice, 153–155; [Solutions of problems in algebra and analytic geometry], 155–162, 167–173—No. 8, May: "Note sur les déterminants" by E. Pouget, 177–183; Questions, Ecole Normale Supérieure et Bourses de Licence, concours spécial pour les démobilisés de 1919," 184–187; [Solutions of problems in algebra and analysis, analytic and descriptive geometry], 187–196.

Revue Générale des Sciences, volume 31, April 15, 1920: "Les bases de la théorie de la relativité" by E. Guillaume, 200–210—May 15: [description of F. E. Wright's [April] article in the *Journal of the Washington Academy of Sciences*, see above], 266; Review by E. Cartan of Veblen and Young's *Projective Geometry* (Boston, 1916–17), 286.

Revue Scientifique, volume 58, April 24, 1920: "Le bicentenaire de John Flamsteed" by E. Doublet, 237–240.

School Science and Mathematics, volume 20, no. 5, May, 1920: "Calculation by geometry of astronomical distances" by J. V. Collins, 416–418; "The character of the roots of a quadratic equation" by R. E. Moritz, 433–434; "Plus and minus signs in algebra" by J. A. Nyberg, 435–437; "The claims of mathematics as a factor in education" by C. N. Moore, 438–442; "Some applications of the project method in high school mathematics" by Edith S. Eaton, 443–447; "The math quest" by Helen Whitaker, 457–459; "The Theorem of Nicomachus" by O. Schmiedel, 462–465; Problem department, 465–467—No. 6, June: "Talk on logarithms and slide rules" by F. Cajori, 527–530; "Freshman college mathematics" by E. E. Watson [Chairman of a committee of the Iowa Association of Mathematics Teachers], 531–538; "Time, rate and distance problems" by J. A. Nyberg, 539–544; "Some applications of the various forms of zero and unity" by R. Morris, 544–548; Problem department, 554–556.

Scientia, volume 27, February, 1920: Review by G. Loria of T. R. Running's *Empirical Formulas* (New York, 1917), 159–161.

Sepiad, Brown University, volume 20, May, 1920: "Mathematical nightmare" by Marion E. Stark, 32–33 [First two of five stanzas:

"Quite oft when I slumber by night-time,
But not when I slumber by day,
I find myself deep in a woodland
Where tiny lost triangles stray.

Hyperboloids, one or two-sheeted,
Like ghosts flit about me in pairs;
A catenoid mews in the branches,
And helices creep from their lairs."]

Texas Mathematics Teachers' Bulletin, volume 5, no. 3, May 10, 1920: "Looking forward," by J. W. Calhoun, 5–7; "A proposed statistical study of high school grades" by E. L. Dodd, 8–10; "The teaching of elementary algebra" by H. J. Ettlinger, 11–13; "Mathematics in the [University of Texas] summer school," 14–18; "The reorganization of the first courses in secondary school mathematics," 19–33 [a preliminary report by the National Committee of Mathematical Requirements including a letter by J. W. Young]; "The straight edge," 34.

Unterrichtsblätter für Mathematik und Naturwissenschaften, volume 26, nos. 1–2, February, 1920: "Ueber die Verwendung der Logarithmentafel im Schulunterricht" by R. Neuendorff, 12–13; "Beziehungen zwischen dem Dreieck, seinem Ankreis- und Fusspunktdreieck und ihre Anwendung zur Herleitung von Formeln für die trigonometrischen Funktionen der Dreieckswinkel" by S. Frühling, 13–18; "Ueber das einem gegebenen Kreissegment eingeschriebene grösste Rechteck und sein räumliches Analogen" by W. Gaedecke, 18–20.

Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht, volume 51, nos. 2–3, March 30, 1920: "Eine neue Definition der stetigen Teilung" by O. Zander,

41–45; "Die Perspektivität im geometrischen Unterricht der OII" by M. Enders, 46–51; "Die Kegelschnitte als Kreisprojektion" by J. Arneberg, 52–57; "Die Exponentialfunktion im Unterricht" by E. Götting, 58–62; "Kleine Mitteilungen," 62–65; "Aufgaben-repertorium," 65–71; "Bücher Besprechungen," 71–87 [not all of the books are mathematical.]

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY U. G. MITCHELL, University of Kansas, Lawrence.

CLUB ACTIVITIES.

The attention of our readers is called to the fact that among the clubs reported this month are two which have not before been listed by the Department. One of these, "The Pascal Circle" of Trinity College, Washington, D. C., has been organized for four years. The other, "The Square" of Washington Square College, New York University, is, so far as the writer is aware, the most recently organized club having begun its activities in February of this year.

MATHEMATICS CLUB OF COLUMBIA UNIVERSITY, New York, N. Y.

[*1918*, 227–228; *1919*, 262.]

Officers 1919–1920: President, Joseph Feldt '21; vice-president, Barclay V. Heuill '21; secretary, Albert E. Meder, Jr., '22; faculty adviser, Professor Lewis P. Siceloff. These officers constituted the program committee.

The average attendance at meetings during the year was twenty-five. The programs were as follows:

October 31, 1919: "Geometric inversions" by Albert E. Meder '22.
November 21: "Cardinal numbers" by Joseph Feldt '21.
December 5: "The catenary and the suspension bridge" by Barclay V. Heuill '21.
December 17: "The Newtonian theory" by Israel Koral '20.
January 9, 1920: "What is a point?" by Professor Cassius J. Keyser.
February 13: "The history of mathematics in Columbia University" by Professor Thomas S. Fiske, chairman of the department of mathematics.
February 27: "How tables of logarithms are made" by Edward H. Reimer '22.
March 12: "The Einstein theory" by Doctor Kenneth W. Lamson of Barnard College.
March 26: "Origins of the calculus" by Oscar Bodansky '22.
April 9: "Famous problems in the history of mathematics" by Professor David Eugene Smith, president of the Mathematical Association of America.
April 21: "The number π" by Frank Frink, Jr., '22.
May 7: "The geometry of the cubic equation" by Professor Henry B. Mitchell.

THE PASCAL CIRCLE, TRINITY COLLEGE, Washington, D. C.

An undergraduate Mathematics Club was organized at Trinity College, Washington, D. C., under the name of the "Pascal Circle," in 1916, to give

students of mathematics an insight into the problems of current magazines and to study the principles underlying mathematical recreations. The Circle had twenty-two names on its roll that year and held fortnightly meetings. The club has improved each year in numbers, spirit and aims. During 1919–20 it had about thirty-five members and included on its program papers on the history of mathematics. The meetings have been informal in character and enlivened by discussions on present day topics or mathematical puzzles and recreations of interest.

The officers of the Circle for the year 1919–20 were: President, Mary C. Duncan '20; vice-president, Margaret M. Walsh '21; Secretary-treasurer, Julia Thomas '22; faculty adviser and honorary president, Professor Marie C. Mangold.

October, 1919: The initial meeting was devoted to the reception of new members and to a short interpretation of the constitution of the Circle by the president. At this meeting it was decided to have special pins for the society and all members were invited to submit designs.

November: "Blaise Pascal" by Mary C. Duncan '20. The speaker explained why the mathematical club of Trinity College chose the name of "Pascal Circle." The mathematical principle underlying "magic number cards" was discussed by the club.

December: "The history of arithmetic" by Margaret Walsh '21. Designs of pins were submitted and that of Margaret Sheehan '19 was accepted. The pin is of black enamel with gold edge and gold letters. It is in the form of a Pascal triangle bearing the Greek letters $\pi\tau\kappa$ representing the initial letters of "Pascal" Trinity College.

February, 1920: "The History of Geometry" by Professor Marie C. Mangold.

March: "A psychological test" by Margaret Sheehan '19: "A psychological problem" by Marguerite Hopper '22.

At a second meeting in this month a paper was read on "The history of algebra" by Julia Thomas '22 and the formation of magic squares was discussed and explained by various members of the club.

April: "Women who have played a part in the history of mathematics" by Margaret Crotty '21.

May: "The church and mathematics" by Catherine Manion '20.

The last meeting of the year was a social meeting.

THE SQUARE, WASHINGTON SQUARE COLLEGE, NEW YORK UNIVERSITY, N. Y.

The Mathematics Club of Washington Square College was organized February 18, 1920, and named the "Square." For the present it was decided not to limit the membership to advanced students and as a consequence a great deal of interest has been shown by freshmen.

The officers for 1920 were: President, Sophie Epstein '23; secretary-treasurer, Rose Beckenstein '23; faculty adviser, Assistant Professor Earnest J. Oglesby.

The average attendance has been about fifteen. Meetings are held twice each month during the academic year.

The following programs were given during the first half of 1920:
February 18, 1920: Organization.
February 26: "Magic squares" by Sophie Epstein '23.
March 11: "Flatland" by Bertha Turetzky '23.
March 25: "Mathematics of warfare" by Isadore Jacobs '23.
April 9: "Π" by Elias Gastman '23.
April 21: "Card tricks" by Isadore Saperstein '23.
May 5: "The duplication of the cube" by Winifred Jacobs '23.

The Mathematics Club of Vassar College, Poughkeepsie, N. Y.

[*1918*, 136, 456; *1919*, 264; *1920*, 78–79.]

The descriptions of the meetings held during 1919–20 were furnished by the secretary, Miss Elizabeth Larsen.

October 28, 1919: This first meeting was a Welcome Picnic to the new members of the club. "Hot-dogs," cider and doughnuts were consumed around a bon-fire. Cards were circulated having fortunes written on them with invisible ink, in the form of milk, which became legible upon being held over the fire. Mathematical charades were acted out by members as part of the evening's amusements.

November 18: This was the first regular meeting of the year and a new plan was presented for conducting the meetings. A different committee was to be appointed to take charge of the program of each meeting. The aim of this scheme was to divide the work and so enlist the interest of a larger number of members. Also, a mathematical contest of problems was announced. Each month problems were to be presented to the club and at the end of the year a prize of $5 was to be awarded to the person solving the greatest number. The subject of the meeting was "The relation of mathematics to science." Talks were given showing the part played by mathematics in physics and chemistry.

December 9: The subject for discussion was "Mathematics and art." It was explained how certain mathematical ideas and principles were present in nature, and the principle of symmetry, the law of order and the existence of a definite proportion known as "the golden section" were illustrated.

January 15, 1920: The subjects for discussion were "The squaring of the circle" and "The duplication of the cube."

February 19: Election of officers and social meeting. Officers elected for the second semester were : President, Mildred Booth '20; vice-president, Amy Davison '21; secretary-treasurer, Elizabeth Larsen '20; faculty member of the executive committee, Professor Elizabeth B. Cowley; student member of executive committee, Helen Lewis '20.

The entertainment program consisted in guessing the pictures of leading

mathematicians and listening to delightful talks on Descartes, Newton and Euclid as seen in an interview.

March 18: Subjects for discussion were "Cranks and their follies" and "The eternal triangle." Part of the hour was spent in solving problems, a prize being awarded for the greatest number of correct solutions. After the business meeting refreshments were served.

April 27: The subject discussed was "The circle" including the nine-point circle and the problem of Apollonius, to construct a circle tangent to each of three given circles. A lighter supplement was added to the purely instructive part of the program by the reading of one of Stephen Leacock's essays entitled "An interview with our greatest scientist."

May 27: Spring picnic. The five-dollar prize for the greatest number of solutions to the problems presented during the year was awarded by Professor Cowley. Various members of the club had composed mathematical parodies of well-known songs and the entertainment feature of the meeting consisted in the singing of these parodies.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about problems and solutions to B. F FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. The editorial work of this department would be greatly facilitated if, on sending in problems, the proposers would also enclose their solutions—*when they have them.* If a problem proposed is not original the proposer is requested *invariably* to state the fact and to give an exact reference to the source.]

2858. Proposed by C. P. SOUSLEY, Pennsylvania State College.

A boy can split wood as fast as his father can saw, and the father can split twice as fast as the son can saw. How should the money received for their labor be divided?

2859. Proposed by L. S. DEDERICK, U. S. Naval Academy.

Derive an expression for the limit of error in evaluating a definite integral by Simpson's Rule.

2860. Proposed by E. O. BROWN, Chicago, Ill.

A frustrum of a right circular cone has a volume v. The lateral area added to the lesser base is a sum which is a minimum. Determine the dimensions of the frustrum in terms of v.

2861. Proposed by B. F. FINKEL, Drury College.

Obtain by plane geometry, *i.e.*, without use of calculus, a construction for finding points on the envelope of a system of circles whose diameters are chords of a fixed circle passing through a given point on it. Also determine geometrically the nature of the locus.

2862. Proposed by J. L. RILEY, Stephensville, Texas.

Show that the whole area commanded by a gun on a hillside is an ellipse whose focus is at the gun, whose eccentricity is the sine of the inclination of the hill to the horizon, and whose semi-latus rectum is twice the greatest height to which the gun could send a ball.

SOLUTIONS.

129 (Average and probabilities) **[1902, 148; 1903, 81]. Proposed by J. K. ELLWOOD, Pittsburg, Pa.**

A and *B* play with two dice, *A* throwing. If he throws 7 or 11, he wins; if he throws 3, or two aces, or two sixes, *B* wins. But if he throws 4, 5, 6, 8, 9, or 10, he continues throwing to duplicate this throw, in which event he wins; if in throwing, however, he throws 7, *B* wins. What is the expectancy of each? [This is the regulation "crap" game, *B* being the banker.]

155 (Average and probabilities) **[1905, 76]. Proposed by E. B. WILSON, Yale University.**

The game of craps is played with two dice. If the player throws 7 or 11 on the first throw he wins. If he throws 12, 2, or 3 he loses. If the player throws any other number, that is to say, 4, 5, 6, 8, 9, 10, he is obliged to continue throwing until he throws that number again or until he throws 7. If he succeeds in throwing his first throw before he does 7, he wins—otherwise he loses. Required the odds against him. (Note that he can continue throwing indefinitely without getting either his original throw or the 7).

NOTE BY R. C. ARCHIBALD, Brown University.

The answers to these problems may be found in the article by Mr. Bancroft H. Brown ("Probabilities in the game of 'shooting craps'") in this MONTHLY, *1919*, 351–352; see also *1920*, 166–167.

2752 [1919, 72]. Proposed by the late R. E. MOORE.

Test for convergence, the series $\sum_{n=1}^{\infty} a_n$, in which

$$a_n = \left[\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\right]^2.$$

I. SOLUTION BY P. J. DA CUNHA, University of Lisbon.

On sait que lorsque le rapport a_n/a_{n+1} est développable suivant les puissances entières de $1/n$, il est très facile de décider, dans tous les cas, s'il y a ou non convergence. En effet, si l' on pose, en s'arrêtant aux termes du second ordre,

$$\frac{a_n}{a_{n+1}} = \alpha + \frac{\beta}{n} + \frac{\theta_n}{n^2},$$

θ_n restant fini pour $n = \infty$, il y a:

Divergence	si $\alpha < 1$ ou $\alpha = 1$ et $\beta \leqq 1$;
Convergence	si $\alpha > 1$ ou $\alpha = 1$ et $\beta > 1$

(Jordan, *Cours d'Analyse*, troisième édition, tome 1, page 313.)

Cela posé, en appliquant la règle à la série donnée, nous avons

$$\frac{a_n}{a_{n+1}} = \left(\frac{2n+2}{2n+1}\right)^2 = \frac{4n^2+8n+4}{4n^2+4n+1} = \frac{1+\frac{2}{n}+\frac{1}{n^2}}{1+\frac{1}{n}+\frac{1}{4n^2}}$$

ou, finalement

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{\theta_n}{n^2}.$$

Comme nous trouvons $\alpha = 1$, $\beta = 1$, la serie considerée est divergente.

II. SOLUTION BY OTTO DUNKEL, Washington University.

A proof of the same nature as that of Professor Cunha but requiring only elementary facts is as follows.

Omit the first factor $(\frac{1}{2})^2$ of each term and consider the series whose general term is

$$b_n = 4a_n = \left(\frac{3}{4}\right)^2\left(\frac{5}{6}\right)^2\cdots\left(\frac{2n-3}{2n-2}\right)^2\left(\frac{2n-1}{2n}\right)^2.$$

Since

$$\left(\frac{2n-1}{2n}\right)^2 = \left(1-\frac{1}{2n}\right)^2 = 1 - \frac{1}{n} + \frac{1}{4n^2} > \frac{n-1}{n},$$

we have, by multiplying all such inequalities from $n = 2$ to $n = n$,

$$b_n > \frac{1}{n}$$

and therefore the b series is divergent and hence also the a series.

Also solved by E. H. CLARKE, R. A. JOHNSON, H. L. OLSON, ARTHUR PELLETIER, and the Proposer.

2758 [1919, 124]. Proposed by LEONARD RICHARDSON, University of British Columbia.
Prove that, if r be a positive integer,

$$\int_0^{\pi/2} \frac{\sin (2r+1)\psi}{\sin \psi}\, d\psi = \frac{\pi}{2}.$$

and

$$\int_0^{\pi/2} \frac{\sin 2r\psi}{\sin \psi}\, d\psi = 2\left\{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{r-1}}{2r-1}\right\}.$$

I. SOLUTION BY ELIJAH SWIFT, University of Vermont.

We have the identity

$$\sin n\psi - \sin (n-2)\psi = 2 \sin \psi \cos (n-1)\psi,$$

or

$$\sin n\psi = 2 \sin \psi \cos (n-1)\psi + \sin (n-2)\psi.$$

This yields us an easy reduction formula for the two integrals. For the second,

$$\int_0^{\pi/2} \frac{\sin 2r\psi}{\sin \psi}\, d\psi = 2\int_0^{\pi/2} \cos (2r-1)\psi \cdot d\psi + \int_0^{\pi/2} \frac{\sin (2r-2)\psi}{\sin \psi}\, d\psi$$

$$= 2\,\frac{(-1)^{r-1}}{2r-1} + \int_0^{\pi/2} \frac{\sin (2r-2)\psi}{\sin \psi}\, d\psi.$$

Applying this r times, we reach the indicated result.

For the first integral,

$$\int_0^{\pi/2} \frac{\sin (2r+1)\psi}{\sin \psi}\, d\psi = 2\int^{\pi/2} \cos 2r\psi \cdot d\psi + \int_0^{\pi/2} \frac{\sin (2r-1)\psi}{\sin \psi}\, d\psi.$$

The first of these is 0. Repeating this process r times, the required integral

$$= \int_0^{\pi/2} \frac{\sin \psi}{\sin \psi} \cdot d\psi = \frac{\pi}{2}.$$

Solved similarly by A. M. HARDING and C. C. YEN.

II. SOLUTION BY R. D. BOHANNAN, Ohio State University.

Let $\cos \psi + i \sin \psi = z$ and $\cos \psi - i \sin \psi = 1/z$. Then, for the first integral, we have

$$\frac{\sin (2r+1)\psi}{\sin \psi}\, d\psi = \frac{z^{2r+1} - \dfrac{1}{z^{2r+1}}}{z - \dfrac{1}{z}} \frac{dz}{iz},$$

$$= \left\{ z^{2r} + z^{2r-2} + \cdots + z^2 + 1 + \frac{1}{z^2} + \cdots + \frac{1}{z^{2r}} \right\} \frac{dz}{iz}.$$

The first and last terms, after integration, give

$$\frac{1}{2ri}\left(z^{2r} - \frac{1}{z^{2r}}\right) \quad \text{or,} \quad \frac{1}{r} \sin 2r\psi \Big]_0^{\pi/2},$$

which is zero; likewise, all other pairs, equidistant from the ends. The middle term gives $(\log z)/i$ or ψ, since $z = e^{i\psi}$, and with the given limits, this reduces to $\pi/2$.

Similarly, the second integral comes from

$$\left\{ z^{2r-1} + z^{2r-3} + \cdots z + \frac{1}{z} + \frac{1}{z^3} + \cdots + \frac{1}{z^{2r-1}} \right\} \frac{dz}{iz}.$$

After integration, the first and last terms, give

$$\frac{1}{i(2r-1)} \left\{ z^{2r-1} - \frac{1}{z^{2r-1}} \right\} \quad \text{or} \quad \frac{2}{2r-1} \sin (2r-1)\psi \Big]_0^{\pi/2} = \pm \frac{2}{2r-1}.$$

The result given is obtained by starting with the central terms,

$$\left(z + \frac{1}{z} \right) \frac{dz}{iz}.$$

Also solved similarly by P. J. DA CUNHA, WILLIAM HERBERG, and H. L. OLSON.

2759 [1919, 124]. Proposed by J. L. RILEY, Stephenville, Texas.
Solve the simultaneous functional equations

$$\phi(x+y) = \phi(x) + \frac{\phi(y)\cdot\psi(x)}{1 - \phi(x)\phi(y)},$$

$$\psi(x+y) = \frac{\psi(x)\cdot\psi(y)}{1 - \phi(x)\phi(y)}.$$

I. SOLUTION BY C. F. GUMMER, Kingston, Ont.

When $y = 0$ the equations show that either $\psi(x) = 0$, or $\phi(0) = 0$ and $\psi(0) = 1$. In the former case, $\phi(x)$ is constant. In the latter, the first equation gives

$$\frac{\phi(x+y) - \phi(x)}{y} = \frac{\phi(y) - \phi(0)}{y} \cdot \frac{\psi(x)}{1 - \phi(x)\phi(y)};$$

so that, if we assume that $\phi'(0)$ exists and equals a, we deduce that

$$\phi'(x) = a\psi(x).$$

If further $\psi'(0)$ exists and equals b, the second equation gives

$$\psi'(x) = \{a\phi(x) + b\}\psi(x).$$

Eliminating $\psi(x)$ between the last two equations, and writing $X(x) = a\phi(x) + b$, we get

$$X''(x) = X(x)X'(x);$$

and on integration, since $X(0) = b$ and $X'(0) = a^2$,

$$X(x) = 2A \tan (Ax + B),$$

where $\sqrt{2a^2 - b^2} = 2A$ and $b = 2A \tan B$.

Hence

$$\phi(x) = \sqrt{2} \cdot \frac{\sin Ax}{\cos (Ax + B)}, \qquad \psi(x) = \frac{\cos^2 B}{\cos^2 (Ax + B)}.$$

This solution, however, fails to satisfy the functional equations unless $A = 0$. Hence the only solutions admitting derivatives for $x = 0$ are

$$\phi(x) = c, \quad \psi(x) = 0 \quad \text{and} \quad \phi(x) = 0, \quad \psi(x) = 1.$$

It is apparent that $\phi(x) = 0$, $\psi(x) = a^x$ is a solution.

Possibly the proposer intended to square the denominator on the right of the second equation. With this change and similar work we get, assuming $\phi'(0)$ and $\psi'(0)$ to exist, *either*

$$\phi(x) = c, \qquad \psi(x) = 0$$

or

$$\phi(x) = \frac{\sin Ax}{\cos (Ax + B)}, \qquad \psi(x) = \frac{\cos^2 B}{\cos^2 (Ax + B)},$$

both satisfying the equations.

II. Solution by A. A. Bennett, Baltimore, Md.

Unlike algebraic equations, the first is sufficient to determine both ϕ and ψ, and the general solution of this first equation will be obtained. The second equation as given is inconsistent with the first except for the trivial solutions, $\phi \equiv 0$, $\psi = a^x$.[1] The equations intended were,

$$\phi(x + y) = \phi(x) + \frac{\phi(y)\psi(x)}{1 - \phi(x)\phi(y)}, \tag{1}$$

$$\psi(x + y) = \frac{\psi(x)\psi(y)}{[1 - \phi(x)\phi(y)]^2}, \tag{2}$$

where the second now serves merely to specialize one of the parameters in the solution of (1).

From the symmetry of (1), we have

$$\phi(x) + \frac{\phi(y)\psi(x)}{1 - \phi(x)\phi(y)} = \phi(y) + \frac{\phi(x)\psi(y)}{1 - \phi(x)\phi(y)}.$$

Excluding, for this discussion, the case $\phi(x) \equiv 0$, this reduces to

$$\frac{\psi(x)}{\phi(x)} - \phi(x) - \frac{1}{\phi(x)} = \frac{\psi(y)}{\phi(y)} - \phi(y) - \frac{1}{\phi(y)};$$

hence, each member is a constant, say $2c$, and

$$\psi(x) = 1 + 2c\phi(x) + \phi^2(x).$$

Equation (2) serves merely to determine c as equal to zero.[2]

To obtain the general solution, let us first obtain a suggestive form by examining only continuous solutions with a continuous derivative at the origin. From (1) by putting $y = 0$, one has $\phi(0) = 0$; hence, $\phi(y)/y$ for y approaching zero will have a finite limit, and $[\phi(x + y) - \phi(x)]/y$ approaches $\phi'(x)$ for y approaching zero. Now (1) may be written

$$\frac{\phi(x + y) - \phi(x)}{y} = \frac{\phi(y)}{y} \frac{\psi(x)}{1 - \phi(x)\phi(y)}.$$

Letting y approach zero, this gives

$$\phi'(x) = \phi'(0)\psi(x) \qquad \text{or} \qquad a\psi(x),$$

and, therefore,

$$\phi'(x) = a[1 + 2c\phi(x) + \phi^2(x)].$$

Integrating, we have, as is at once verified

$$\begin{cases} \phi(x) = \sqrt{1 - c^2} \tan [ax\sqrt{1 - c^2} + \text{arc sin } c] - c \\ \psi(x) = (1 - c^2) \sec^2 [ax\sqrt{1 - c^2} + \text{arc sin } c] \end{cases}$$

as the most general solution satisfying the continuity conditions indicated. For $c^2 > 1$, the expression contains imaginary terms but a real solution in logarithmic form may be given as is customary in differential equations. These are, however, algebraically equivalent.

To obtain the most general solution where ϕ is not identically zero, put

$$\phi(x) = \sqrt{1 - c^2} \tan [a\sqrt{1 - c^2} f(x) + \text{arc sin } c] - c$$

in (1). This determines $\psi(x)$ as $(1 - c^2) \sec^2 [a\sqrt{1 - c^2} \cdot f(x) + \text{arc sin } c]$. Equation (1) now reduces to $af(x + y) = af(x) + af(y)$. This is Cauchy's first functional equation of which the continuous solutions are $af(x) = ax$ except for a change of parameter, and of which the discontinuous solutions are all known for purposes of functional equations. For $c = 0$, the case

[1] The solution $\phi(x) = \text{const.}$, $\psi(x) \equiv 0$, is overlooked.—Editors.
probably,

[2] The reasoning here calls for the emendation in footnote 1.—Editors.

when both (1) and (2) are satisfied, one has

$$\phi(x) = \tan ax, \qquad \psi(x) = \sec^2 ax,$$

which is the most general continuous solution, [also $\psi(x) \equiv 0$ and $\phi(x) = \text{const.}$—EDS.].

Also solved by ELIJAH SWIFT; partial solutions by E. B. ESCOTT, and H.L. OLSON.

2763 [1919, 170] Proposed by C. N. SCHMALL, New York City.

Show that the equation $ky - 2k^{1/3}a^{2/3}x + x^2 = 0$, where k is a variable parameter, represents a family of parabolas passing through a fixed point, and all having the same areas comprised between the curve and the x-axis.

Show, also, that the envelope of the family is the rectangular hyperbola whose equation is $xy = 2^5a^2/3^3$.

SOLUTION BY P. J. DA CUNHA, University of Lisbon.

Il est évident que l'équation donnée représente une famille de paraboles, et que toutes ces courbes jouissent de la propriété de passer par l'origine. L'abscisse du second point d'intersection de chacune des paraboles et de l'axe des x est $2k^{1/3}a^{2/3}$, de sorte que l'aire comprise entre la courbe et le même axe est

$$A = \frac{1}{k}\int_0^{2k^{1/3}a^{2/3}} (2k^{1/3}a^{2/3}x - x^2)dx = \frac{1}{k}\left[k^{1/3}a^{2/3}x^2 - \frac{x^3}{3} \right]_0^{2k^{1/3}a^{2/3}} = \frac{4}{3}a^2.$$

Finalement, l'équation de l'enveloppe s'obtenant par la règle connue, c'est à dire, en éliminant k entre l'équation donnée et celle-ci: $y - 2k^{-2/3}a^{2/3}x/3 = 0$, nous tombons sur l'équation

$$xy = \frac{2^5a^2}{3^3},$$

c. q. f. d.

After giving this same solution H. H. Downing, University of Kentucky, notes that the locus of the vertices of the parabolas is the equilateral hyperbola, $x\,y = a^2$.

The problem was also solved by C. A. BARNHART, A. M. HARDING, WILLIAM HERBERG, WILLIAM HOOVER, ARTHUR PELLETIER, and J. B. REYNOLDS.

2766 [1919, 171]. Proposed by N. P. PANDYA, Amreli, India.

Is it possible to find a harmonic series whose terms are positive integers such that the product of the first, second, seventh, and eighth terms is equal to the product of the third, fourth, fifth, and sixth terms?

SOLUTION BY A. M. HARDING, University of Arkansas.

Any harmonic series of eight terms may be written in the form

$$a, \quad \frac{a}{1+ak}, \quad \frac{a}{1+2ak}, \quad \cdots, \quad \frac{a}{1+7ak},$$

where a and k are constants, $a \neq 0$. If it is possible to find an harmonic series satisfying the conditions of the problem, then

$$a \cdot \frac{a}{1+ak} \cdot \frac{a}{1+6ak} \cdot \frac{a}{1+7ak} = \frac{a}{1+2ak} \cdot \frac{a}{1+3ak} \cdot \frac{a}{1+4ak} \cdot \frac{a}{1+5ak}.$$

That is, since $a \neq 0$,

$$(1+ak)(1+6ak)(1+7ak) = (1+2ak)(1+3ak)(1+4ak)(1+5ak),$$

or

$$16a^2k^2 + 112a^3k^3 + 120a^4k^4 = 0.$$

Supposing $k \neq 0$ and dividing by $8a^2k^2$, we have

$$2 + 14ak + 15a^2k^2 = 0;$$

whence

$$ak = \frac{-7 \pm \sqrt{19}}{15}.$$

Then

$$1 + ak = \frac{8 \pm \sqrt{19}}{15}, \quad \text{and} \quad \frac{a}{1 + ak} = \frac{15a}{8 \pm \sqrt{19}}.$$

That is, if a is a positive integer, the second term of the series is irrational. Hence, the only series which satisfies the condition of the problem is one for which $k = 0$; that is, a series in which the terms are all equal.

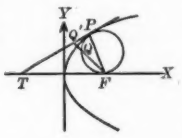
Also solved by H. L. Olson and Arthur Pelletier.

2779 [1919, 268]. Proposed by J. L. Riley, Junior Agricultural and Mechanics College, Stephensville, Texas.

A parabola is placed with its axis horizontal; find the straight line of shortest descent from the curve to the focus.

I. Solution by A. M. Harding, University of Arkansas.

Construct a circle tangent to the axis of the parabola at F and tangent to the parabola at P[1]. The time of descent down all chords of this circle which pass through F is the same, so that the time down PF is the same as the time down any other chord QF, and is, therefore, less than the time down the focal radius $Q'F$.



II. Solution by H. S. Uhler, Yale University.

Let $p \equiv$ distance from vertex to focus, $g \equiv$ acceleration due to gravity, $s \equiv$ distance from any point (x, y) on the parabola and in the first quadrant to the focus $(p, 0)$, $t \equiv$ time of descent, and $\gamma \equiv$ angle which the straight line makes with the positive direction of the axis of x. From kinematics, $s = \frac{1}{2}at^2$ and $a = g \sin \gamma$; hence,

$$t^2 = \frac{2s}{g \sin \gamma}. \tag{1}$$

The polar equation of the parabola, with the pole at the focus, is

$$s = \frac{2p}{1 - \cos \gamma} = \frac{2p(1 + \cos \gamma)}{\sin^2 \gamma}. \tag{2}$$

Hence, from (1) and (2),

$$\frac{g}{4p} t^2 = \frac{1 + \cos \gamma}{\sin^3 \gamma}.$$

Differentiating,

$$\frac{gt}{8p} \cdot \frac{dt}{d\gamma} = \frac{-\sin^2 \gamma - 3(1 + \cos \gamma) \cos \gamma}{\sin^4 \gamma} = 0;$$

therefore,

$$2 \cos^2 \gamma + 3 \cos \gamma + 1 = 0$$

so that

$$\cos \gamma = -1 \quad \text{or} \quad \cos \gamma = -\tfrac{1}{2}.$$

[1] If this construction is possible and if the tangent at P meets the axis in T, $TP = TF = x + p$, the equation of the parabola being $y^2 = 4px$. Hence to get the coördinates of P we have $(x + p)^2 = 4x^2 + 4px$ or $x = \frac{1}{3}p$. It follows that the triangle TPF is an equilateral triangle, each side equal to $\frac{4}{3}p$. The coördinates of the center of the circle will be p, $4p/3\sqrt{3}$, and its equation $(x - p)^2 + y^2 - 8py/3\sqrt{3} = 0$. If we eliminate x or y from this equation and the equation of the parabola we shallfind for the resulting equation two equal roots corresponding to p, and two complex roots, showing that the circle must lie entirely within the parabola—which is an important consideration in connection with Professor Harding's discussion of the problem.—Editors.

An examination of the factored form of the derivative shows that $\gamma = 120°$ gives a minimum.

Incidentally,

$$t = 4\sqrt{\frac{p}{3g\sqrt{3}}}, \qquad s = \tfrac{4}{3}p.$$

Also solved by T. M. BLAKSLEE, R. A. JOHNSON, H. L. OLSON, ARTHUR PELLETIER, S. W. REAVES, J. B. REYNOLDS, and ELIJAH SWIFT.

2794 [1919 458]. Proposed by B. J. BROWN, Kansas City, Mo.

Find the value of $x^{e^x} \div x^{x^x}$ when $x \doteq 0$ and when $x \doteq \infty$. I. C. S. 1902.

SOLUTION BY PAUL CAPRON, U. S. Naval Academy.

A^{b^c} is taken to mean $A^{(b^c)}$.

(i) When $x \doteq 0$, it is well known that $e^x \doteq 1$, $x^x \doteq 1$;

$$x(\log x)^n \doteq \frac{(\log x)^n}{1/x} \doteq -n\frac{(\log x)^{n-1}}{1/x} \doteq \cdots \doteq \pm n!\, x \doteq 0. \qquad y = x^{e^x} \div x^{x^x} = x^{e^x - x^x}.$$

$$\log y = (e^x - x^x)\log x = \frac{e^x - x^x}{1/\log x} \doteq \frac{0}{0} \doteq \frac{e^x - x^x(1+\log x)}{-\dfrac{1}{x(\log x)^2}}$$

$$\doteq -xe^x(\log x)^2 + x^x(x(\log x)^2 + x(\log x)^2 \doteq 1.0 + 1(0+0) = 0.$$

Hence, as $x \doteq 0$, $y \doteq 1$.

(ii) When $x \doteq \infty$,

$$x^x \div e^x = \left(\frac{x}{e}\right)^x \doteq \infty;$$

hence,

$$e^x - x^x = e^x\left(1 - \left(\frac{x}{e}\right)^x\right) \doteq -\infty;$$

i.e., with the notation of (i), $\log y = (e^x - x^x)\log x \doteq (-\infty)(+\infty) \doteq -\infty$, or $y \doteq 0$.

2795 [1919, 458]. Proposed by C. N. SCHMALL, New York City.

A square is described touching the ellipse, $x^2/a^2 + y^2/b^2 = 1$, at the ends of its minor axis; a second ellipse is drawn circumscribing the square and tangent to the given ellipse at the ends of the major axis. The new ellipse is treated as the first and the process is continued until there are n new ellipses. Show that the last ellipse is a circle if the eccentricity of the original ellipse is $\sqrt{n/(n+1)}$.

SOLUTION BY GERTRUDE I. MCCAIN, Oxford, Ohio.

If x', y' be the coördinates of a corner of the first square, then each equals b; and, lying on the second ellipse,

$$\frac{x'^2}{a^2} + \frac{y'^2}{b_1^2} = 1,$$

where b_1 is the minor axis of the first circumscribed ellipse. Substituting b for x' and y' and solving for b_1^2,

$$b_1^2 = \frac{a^2b^2}{a^2 - b^2}.$$

Similarly,

$$b_2^2 = \frac{a^2b_1^2}{a^2 - b_1^2} = \frac{a^2b^2}{a^2 - 2b^2},$$

and

$$b_n^2 = \frac{a^2b^2}{a^2 - nb^2}.$$

If $\frac{a^2 - b^2}{a^2} = \frac{n}{n+1}$, then $n = \frac{a^2 - b^2}{b^2}$. Substituting this value of n, $b_n^2 = a^2$. Hence, the last ellipse is a circle.

Also solved by NORMAN ANNING, GRACE M. BAREIS, PAUL CAPRON, H. O. HANSON, OLIVE C. HAZLETT, E. J. OGLESBY, H. L. OLSON, ARTHUR PELLETIER, J. L. RILEY, C. P. SOUSLEY, ELIJAH SWIFT, and H. S. UHLER.

NOTES AND NEWS.

IT IS HOPED THAT READERS OF THE MONTHLY WILL COOPERATE IN CONTRIBUTING TO THE GENERAL INTEREST OF THIS DEPARTMENT BY SENDING ITEMS TO THE EDITOR-IN-CHIEF.

Mr. L. M. GRAVES of Chicago, has been appointed instructor in mathematics at Washington University.

Miss LOUISA M. WEBSTER, associate professor of mathematics at Hunter College, has been appointed principal of the Hunter College High School.

Mr. H. L. OLSON, of the University of Wisconsin, has been appointed instructor in mathematics at the University of Michigan.

At the University of Nebraska, Dr. T. A. PIERCE has been promoted to an assistant-professorship of mathematics; Mr. W. M. BOND and Mr. C. R. SHERER have been appointed instructors, and Miss C. RUMMONS assistant instructor.

Mr. OSCAR SCHMIEDEL has been appointed professor of mathematics at Nebraska Wesleyan University.

H. S. MYERS, for four years professor of mathematics at Huron (S. D.) College, has taken a similar position at Southwestern College, Winfield, Kansas.

Professor SOLOMON LEFSCHETZ of the University of Kansas will be absent on leave during this academic year most of which he will spend in Europe.

Professor HARRIET GLAZIER of Western College for Women has been given an extended leave of absence for a second year and will spend the year at Southern Branch of the University of California, filling the vacancy caused by the absence for the year of Professor MYRTIE COLLIER.

At the University of Texas Miss ANNA MULLIGAN and Miss HELMA HOLMES have been appointed instructors, and Mr. CLAUDE BAILEY, tutor, in mathematics.

Mr. O. C. COLLINS, of Oxford University, England, has been appointed instructor in mathematics at the University of Nebraska.

Mr. C. E. HARRINGTON, M. E. (Cornell), has been appointed as an instructor in mathematics at the University of Buffalo.

Professor E. H. THOMAS of Tabor College has been appointed professor of physics at Southeast Missouri State Teachers College.

Mr. J. B. ROSENBACH, last year at the University of Illinois, has been appointed to an instructorship in mathematics at the Carnegie Institute of Technology. During the summer vacation Mr. Rosenbach was employed in the valuation department of the Atchison, Topeka and Santa Fe Railway.

Miss CLARIBEL KENDALL of the University of Colorado will spend her year's leave of absence at the University of Chicago as Fellow in mathematics, her place being supplied by W. H. HILL of Greeley (Colo.) High School.

Dr. E. G. BILL has been promoted to a full professorship of mathematics at Dartmouth College.

Dr. O. J. RAMLER has been promoted to be associate professor of mathematics at the Catholic University of America.

Mr. F. V. MORLEY, of Johns Hopkins University, has left for study at Oxford University where he enters as a Rhodes Scholar.

Dr. DANIEL BUCHANAN, professor of astronomy and mathematics at Queen's University, has been appointed professor of mathematics and head of the department at the University of British Columbia.

In the department of mathematics at the University of Minnesota Associate Professor R. W. BRINK [*1920*, 382], who has been lecturer at the University of Edinburgh during 1919–20, has resumed his work. Mr. C. M. JENSEN, of the University of Minnesota, and Miss RUTH ASKELAND, of Hamline College, have been appointed teaching fellows; and Miss ELIZABETH CARLSON, of the University of Minnesota, a teaching assistant.

At the University of Wyoming Professor C. B. RIDGAWAY is retiring after twenty-four years' service; Professor C. E. STROMQUIST has been made head of the department; A. R. FEHN of Kansas Agricultural College has been made associate professor of mathematics, and LUCY A. FEDDERSEN has been appointed instructor in mathematics in the University High School.

At the University of Pennsylvania Assistant Professor F. H. SAFFORD has been promoted to a full professorship of mathematics; Dr. R. A. ARMS has resigned his instructorship to accept a professorship at Pennsylvania College; and Mr. H. M. GEHMAN and R. W. HARTLEY have been appointed instructors in mathematics.

At the University of Alberta, Lecturer GEORGE ROBINSON has been appointed assistant to Professor E. T. WHITTAKER in his mathematical laboratory in Edinburgh; Mr. T. H. MILNE, of the department of physics at the University of Toronto has been appointed lecturer in mathematics; Mrs. E. T. MITCHELL, of the University of Alberta, has been appointed instructor.

At the University of California, Dr. A. R. WILLIAMS, for three years instructor in the Shipping Board at Portland, Oregon, and Dr. C. D. SHANE have been appointed instructors; Mrs. A. D. B. SHANE, Ph.D., Miss NINA ALDERTON, Miss G. M. CAMPBELL, and Messrs. P. H. DAUS, B. C. WONG, VICTOR STEED and J. F. POBANZ have been appointed assistants.

At the University of Saskatchewan, Professor L. L. DINES [*1919*, 84] is acting head of the department of mathematics while Professor G. H. LING is spending a year in Europe. Dr. I. A. BARNETT has been appointed assistant professor of mathematics, having been released from his acceptance of an instructorship at the University of Illinois [*1920*, 190].

In the department of mathematics at the University of Illinois, Associate

Professor R. D. CARMICHAEL has been promoted to be a professor and Drs. L. L. STEIMLEY and C. F. GREEN have been advanced from assistantships to instructorships; Drs. B. MARGARET TURNER and MARGARET HASEMAN, of Bryn Mawr, have been appointed instructors; Associate Professor E. R. SMITH, of Pennsylvania State College, has leave of absence for a year to serve as an associate; Messrs. J. W. WAGNER, of Baldwin Wallace College, H. A. BENDER, instructor in Muskingum College, L. H. MCFARLANE, of the University of Missouri, R. F. GRAESSER and J. W. HEARST, of the University of Illinois, and Mrs. FENNER STICKNEY, of the University of California, have been appointed assistants.

Dr. EINAR HILLE, of the University of Stockholm, is studying mathematics at Harvard University as a fellow of the Swedish-American Foundation.

ERIC DOOLITTLE, professor of astronomy in the University of Pennsylvania and director of the Flower Observatory since 1912, died on September 21, 1920, aged sixty-one years. He was instructor in astronomy at the University of Pennsylvania 1896–1904, and assistant professor 1904–1912. We have already referred [*1919*, 178] to the death of his father, Professor C. L. DOOLITTLE, his predecessor as director of the Flower Observatory.

T. R. DAVIES, assistant professor of mathematics at McGill University since 1909, died on August 19, 1920, aged fifty-six years. He was born in England and graduated at the University of Cambridge. For several years he was headmaster of Abingdon school, Montreal. In 1907 he was appointed lecturer in mathematics at McGill.

K. H. STRUVE, director of the Berlin-Babelsberg Observatory, and professor of astronomy in the University of Berlin since 1904, died August 12, 1920, aged sixty-six years. Professor Struve was a Russian by birth. His father O. W. STRUVE (1819–1905) and grandfather F. G. W. STRUVE (1793–1864) were also astronomers—working for many years in Russian observatories.

Sir J. N. LOCKYER died August 16, 1920, aged eighty-four years. He was the founder of *Nature*, director of the Hill Observatory, Salcombe Regis, Sidmouth, England, and late director of the solar physics observatory, South Kensington, 1885–1913. For sketches of his life and work see *Nature*, August 19, 26, and September 2, 1920.

Dr. GIOVANNI CELORIA, director of the observatory of Brera, Milan, since 1900, and author of numerous books and papers on astronomical subjects, died August 17, 1920, aged 78 years.

Dr. P. G. H. BACHMANN, professor at the University of Münster from 1875 till made professor emeritus in 1900, died on March 31, 1920, aged eighty-three years. He wrote many books and papers in the field of theory of numbers, and was the author of the sections on "Analytische Zahlentheorie," 1900, and "Niedere Zahlentheorie," 1900 (French editions 1906–1910), in the *Encyklopädie der mathematischen Wissenschaften*. His most recent book was *Das Fermatproblem in seiner bisherigen Entwicklung*, 1919. Among his earlier works were the following: *Die Lehre von der Kreistheilung und ihre Beziehung zur Zahlentheorie*,

1872 (translated into Italian); *Die Elemente der Zahlentheorie*, 1892; *Die analytische Zahlentheorie*, 1894; *Die Arithmetik der quadratischen formen*, 1898; *Niedere Zahlentheorie*, 1902–1910; *Allgemeine Arithmetik der Zahlenkorper*, 1905; and *Grundlehren der neueren Zahlentheorie*, 1907.

JOHN PERRY, emeritus professor of mechanics and mathematics, Royal College of Science, South Kensington, London, died August 4, 1920, aged seventy years. He was born in Ulster and educated at Queens College, Belfast. From 1875 to 1879 he was professor of engineering in Japan, and from 1881–96 professor of engineering and mathematics at the City and Guilds of London Technical College, Finsbury. He was the author of many books and scores of papers; among the former are: *Practical Mechanics*, 1883; *Spinning Tops*, 1890 (2nd German ed. 1913); *Calculus for Engineers*, 1897 (German ed. 1902, 2nd ed. 1910; Russian ed. 1904); *Applied mechanics*, 1897 (French ed. 1913; German ed. 1908); *Practical Mathematics*, 1899, 1907, 1910 (Russian ed. 1909); and *Elementary Practical Mathematics*, 1913. There were many other translations of his books into foreign languages. Perry's *England's Neglect of Science*, 1900, and *Discussion on the Teaching of Mathematics* (which he edited), 1901, contain arguments against the culture value of mathematics—the basis of the "Perry movement" (Cf. *Report of the National Committee of Fifteen on Geometry Syllabus*, Washington, 1912, pp. 27–28). For a sketch of Perry's career see *Nature*, August 12, 1920.

The twenty-seventh meeting and ninth colloquium of the American Mathematical Society were held at the University of Chicago, September 7–11. One hundred and one members of the Society were present at the meeting and thirty four papers were read. The authors of thirty-two of these papers are also members of the Association; for details see the *Bulletin of the American Mathematical Society*, November, 1920. The attendance of eighty-eight persons at the colloquium was considerably in excess of that at any previous colloquium. Two courses of five lectures each were given "Dynamical systems" by Professor G. D. Birkhoff, and "Topics from the theory of functions of infinitely many variables" by Professor F. R. Moulton.

At the meeting of the American Academy of Arts and Sciences on October 13 the following communications were delivered by members of the Association: "Einstein's first theory of relativity" by C. L. E. Moore; "Einstein's generalized relativity or theory of gravitation" by G. D. Birkhoff.

The so-called International Congress of Mathematicians [*1920*, 191] was held at Strasbourg, September 22–30, 1920. The first and fifth days, and the last two days, were reserved for excursions. There were four general lectures, one by L. E. DICKSON, University of Chicago, on "Relations between the theory of numbers and other branches of mathematics"; the others by Larmor, De la Vallée-Poussin, Volterra, and Nörlund. The papers were read in each of four sections; among the speakers were eight American mathematicians. In Section I (Arithmetic, Algebra, Analysis) there were 33 papers of which the following were by American mathematicians: "Homogeneous polynomials

with a multiplication theorem" by L. E. DICKSON; "On Stieltjes integrals and Volterra compositions" by P. J. DANIELL, Rice Institute; "On certain iterative properties of bilinear operation" and "On the theory of sets of points in terms of continuous transformations" by N. WIENER, Massachusetts Institute of Technology; "Quelques remarques sur la multiplication complexe" by S. LEFSCHETZ, University of Kansas; "On the location of the roots of a polynomial" by J. L. WALSH, Harvard University. In Section II (Geometry) the 13 papers presented included: "La géométrie des variables complexes" by J. S. TAYLOR, Massachusetts Institute of Technology; "Transformation des systèmes conjugués R" and "Transformation des surfaces applicables sur une quadrique" by L. P. EISENHART, Princeton University; "Method of classifying all polygons having a given set of vertices" by F. H. MURRAY, Harvard University. In Section III (Mechanics, mathematical physics, applied mathematics) there were 23 papers; in Section IV (Philosophical, historical and pedagogical questions), 10 papers.

Only three Englishmen took part in the program: Larmor, Young, and Greenhill; Italians, Spaniards, and Scandinavians, as well as Germans and Austrians, were conspicuously absent. The total attendance was above 200. The International Mathematical Union voted to hold the International Mathematical Congress of 1924 in the United States.

The following 16 doctorates with mathematics as major subject were conferred by American universities in the academic year 1919–1920; the title of the dissertation is added in each case: E. M. BERRY, Iowa, "Diffuse reflection"; J. D. BOND, Michigan, "Plane trigonometry in Richard Wallingford's Quadri partium de sinibus demonstratis"; J. DOUGLAS, Columbia, "On certain two-point properties of general families of curves"; T. C. FRY, Wisconsin, "The use of divergent integrals in the solution of differential equations"; GLADYS GIBBENS, Chicago, "Comparison of different line-geometric representations for functions of a complex variable"; C. F. GREEN, Illinois, "On the summability and regions of summability of a general class of series of the form $\Sigma c_n g(x + n)$"; J. W. LASLEY, Chicago, "Some special cases of the flecnode transformation of ruled surfaces"; ELSIE J. MCFARLAND, California, "On a special quartic curve"; J. J. NASSAU, Syracuse, "Some theorems in alternates"; C. A. NELSON, Chicago, "Conjugate systems with conjugate axis curves"; E. L. POST, Columbia, "Introduction to a general theory of elementary propositions"; SUSAN M. RAMBO, Michigan, "The point at infinity as a regular point of certain difference equations of the second order"; L. L. STEIMLEY, Illinois, "On a general class of series of the form $Y(x) = C_0 + \Sigma C_n g(nx)$"; J. L. WALSH, Harvard, "On the location of the roots of the jacobian of two binary forms"; R. WOODS, Illinois, "The elliptic modular functions associated with the elliptic norm curve E^7"; T. YANG, Syracuse, "A problem in differential geometry."

In this MONTHLY, *1920*, 341, a general statement was made concerning the inferiority of the doctorat de l'université, in contrast to the state doctorate, as awarded in France to mathematicians. Professor HARRIS HANCOCK has kindly

sent us a copy of a letter, written in July, 1900 by Professor Gaston Darboux, dean of the faculty of sciences, from which it appears that at the University of Paris an exception should be made to the general statement. The "doctorat de l'université de Paris" is considered by that faculty of sciences as on a par with the state doctorate; Darboux wrote: "les épreuves sont les mêmes, la séverité est la même pour les deux doctorats."

There are several vacancies for the position of computer at the Ballistic Station of the U. S. Ordnance Department, situated in the Coca Cola Bldg., Baltimore, Md. It is hoped to fill these with recent college graduates, either young men or young women who have completed a course in calculus and would like experience in practical computational work for a year at least. Owing to the present urgency and the difficulties experienced, the initial salary offered has been usually $1400. Further particulars may be obtained by addressing the Ballistic Station.

THE WORK OF THE NATIONAL COMMITTEE ON MATHEMATICAL REQUIREMENTS. September 15, 1920 [*1920*, 341–342].

The National Committee on Mathematical Requirements held a meeting at Lake Delavan, Wisconsin, on September 2, 3, and 4 at which a number of reports were discussed and adopted. A report on "The revision of college entrance requirements" received the greatest amount of discussion. It is hoped that this report may be released for publication late in October. It includes a general discussion of the present problems connected with college entrance requirements in mathematics. It is a report of an investigation, recently made by the National Committee, concerning the values of the various topics in elementary algebra as preparation for the elementary college courses in other subjects. It contains also a suggested revision of the definitions of entrance units in elementary algebra and plane geometry. In connection with the suggested requirements in plane geometry a list of fundamental propositions and constructions is attached. This list includes the propositions which may be assumed without proof or given informal treatment, a list of the fundamental theorems and constructions from which it is intended that questions on entrance examination papers other than originals be chosen, and a list of subsidiary theorems. It is proposed to prepare a mimeographed edition of this list of propositions and constructions at the earliest possible moment for the benefit of such teachers as may desire to make use of it in connection with their classes during the coming year. A copy will be sent to any person interested upon application to the Chairman of the Committee (J. W. Young, Hanover, New Hampshire).

A preliminary draft of a report on "Mathematics in experimental schools" was discussed at this meeting. Mr. Raleigh Schorling of the Committee has spent over a year collecting material for this report. It is hoped that it will be ready for publication early next spring. The report will be an extensive one and will describe in detail the work actually done in mathematics in experimental schools throughout the country.

Miss Vevia Blair of the Committee presented her report on the "Present status of disciplinary values in education." It is expected that this report also will be released for publication in October. It gives a critical review of the complete literature concerning the experimental work on the transfer of training as well as an evaluation of this literature terminating in the formulation of certain propositions concerning disciplinary values which appear justified by the experimental work. A particularly valuable feature of the report would seem to lie in the fact that a large majority of the most prominent psychologists in the country appear to be ready to subscribe to the propositions formulated.

Professor E. R. Hedrick presented a report which he prepared at the request of the National Committee on "The function concept in secondary school mathematics." This report also will be published in the near future and is intended ultimately to form a part of the final report of the Committee on the "Reorganization of the first courses in secondary school mathematics." (A preliminary report on this subject was published for the Committee by the U. S. Bureau of Education last February as Secondary School Circular No. 5.)

A preliminary report on "Junior high school mathematics" is in the press of the U. S. Bureau of Education and should be ready for distribution early in October. The National Committee desires the assistance of its cooperating organizations, which now number over 70, in the revision of this preliminary report. Comments, suggestions and criticisms should be sent to the Chairman of the Committee not later than January 1, 1921, in view of the fact that the Committee expects to take up the formulation of its final report on this subject immediately after this date.

A subcommittee under the chairmanship of Professor C. N. Moore is preparing a report on "Elective courses in mathematics in secondary schools." A committee under the chairmanship of Professor David Eugene Smith is preparing a report on "The standardization of terminology and symbolism" and Professor R. C. Archibald is preparing one on "The training of teachers." It is expected that two of these reports will be presented for the consideration of the National Committee in October.

The work of the National Committee and its recommendations were discussed in teachers classes at the summer sessions of colleges, universities and normal schools throughout the country. Addresses on the work of the Committee were given as follows: by Mr. Raleigh Schorling at Harvard University, by Professor E. R. Hedrick at the Universities of Texas and of Oklahoma, and by Mr. J. A. Foberg at the Universities of Iowa and Minnesota.

Present indications point to the fact that the work of the National Committee will have a prominent place on the programs of most teachers organizations throughout the country during the coming year. The National Committee stands ready as before to help in every possible way in the preparation of such programs and will be glad to furnish material for discussion.

It will also be pleased to furnish speakers for such meetings to the extent of its ability.

RETROSPECT AND PROSPECT FOR MATHEMATICS IN AMERICA.

RETIRING PRESIDENTIAL ADDRESS DELIVERED BEFORE THE MATHEMATICAL ASSOCIATION OF AMERICA, SEPTEMBER 66, 1920.

By H. E. SLAUGHT, University of Chicago.

On several occasions since January, 1913, when twelve men, representing as many universities and colleges of the Middle West, took over THE AMERICAN MATHEMATICAL MONTHLY and dedicated it to the interests of collegiate mathematics, editorial comment in this journal has naturally taken the form of gratitude for past successes and optimism for future achievement. One of those occasions[1] was at the end of the first year of this critical undertaking when the subscription list had been doubled and it was becoming apparent that the self-imposed duties and responsibilities of that first editorial board, in their effort to render genuine service to the teachers of collegiate mathematics, were eliciting widespread approbation. Another such occasion[2] was at the end of three years, when the subscription list had again doubled, the subvention being no longer needed had been released, and over one thousand charter members had joined the newly organized MATHEMATICAL ASSOCIATION OF AMERICA with the MONTHLY as its official journal.

It seems highly appropriate now, near the end of the eighth year, that my remarks on the present occasion should again fall into the form of "Retrospect and Prospect," not alone directly with respect to the Association and the MONTHLY, but more especially with respect to recent developments in mathematical activity as a whole in America and the rôle which the Association and the MONTHLY may reasonably be expected to play in future developments.

First, let us recall and emphasize the fundamental things for which the Association and the MONTHLY stand. By the terms of its constitution the object of the Association is "to assist in promoting the interests of mathematics in America, especially in the collegiate field." This is, indeed, a broad charter, but it is subject to one obvious limitation inherent in the very circumstances under which the Association was conceived and organized, namely, that there should be no conflict in ideals between the Association and the American Mathematical Society, whose object, as stated by its constitution, is "to encourage and maintain an active interest in mathematical science." Just as the Society, by the resolution of its Council[3] in April 1915, decided that it was unwise to enter into the activities then represented by THE AMERICAN MATHEMATICAL MONTHLY, but expressed its realization of the importance of work in that field and its

[1] "Retrospect and prospect" by H. E. Slaught, AMERICAN MATHEMATICAL MONTHLY, *1913*, 1–3.

[2] "A tentative platform of the Association" by E. R. Hedrick, AMERICAN MATHEMATICAL MONTHLY, *1916*, 31–33.

[3] *Bulletin of the American Mathematical Society*, Vol. 21, page 482.